Math 2070 Final Exam Solutions (Fall 1994)

1. (Linear Eq).
Integrating factor $= \mu(x) = e^{3x}$. $\Rightarrow e^{3x}y' + 3e^{3x}y = 8e^{5x}$.

$\Rightarrow \frac{d}{dx}(e^{3x}y) = 8e^{5x} \Rightarrow e^{3x}y = \int 8e^{5x}dx = \frac{8}{5}e^{5x} + c$

$\Rightarrow \boxed{y = 8/5\, e^{2x} + ce^{-3x}}$

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2. (Separable Eq). $3y'y^2 = 4x \Rightarrow y^3 = 2x^2 + c$

$\Rightarrow y = (2x^2+c)^{1/3}$

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3. $p(\lambda) = \lambda^2 - 14\lambda + 49 = (\lambda-7)^2 \Rightarrow y = (c_1 + c_2 x)e^{7x}$

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4. $y_p = Ae^{4x} + Bx + C$, $y_p' = 4Ae^{4x} + B$, $y_p'' = 16Ae^{4x}$

$\Rightarrow 5e^{4x} + 6x = y_p'' - y_p' - 2y_p = (16-4-2)Ae^{4x} + (-2B)x + (-B-2C)$

$\Rightarrow 10A = 5,\ 6 = -2B,\ -B-2C = 0 \Rightarrow A = \frac{1}{2},\ B = -3,\ C = \frac{3}{2}$

$\Rightarrow y_p = \frac{1}{2}e^{5x} - 3x + \frac{3}{2}$.

$p(\lambda) = \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1) \Rightarrow y_h = c_1 e^{2x} + c_2 e^{-x}$

$\Rightarrow \boxed{y = c_1 e^{2x} + c_2 e^{-x} + \frac{1}{2}e^{5x} - 3x + \frac{3}{2}}$

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5. $p(\lambda) = \lambda^2 + 6\lambda + 13 = (\lambda+3)^2 + 4 = 0 \Rightarrow \lambda = -3 \pm 2i$.

$\Rightarrow y = c_1 e^{-3x}\cos 2x + c_2 e^{-3x}\sin 2x$

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6. $p(\lambda) = \lambda^4 - 8\lambda^2 + 16 = (\lambda^2-4)^2 = (\lambda-2)^2(\lambda+2)^2$

Roots $= \pm 2$ with mult. 2.

$\Rightarrow y = (c_1 + c_2 x)e^{2x} + (c_3 + c_4 x)e^{-2x}$

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7. Let $Y = L\{y\}$. Then $L\{y''\} = s^2 Y - s + 1$

$\Rightarrow s^2 Y - s + 1 + 16Y = L\{\delta(t-3)\} = e^{-3s}$

$\Rightarrow Y = \frac{s-1}{s^2+16} + \frac{1}{s^2+16}e^{-3s}$.

$L^{-1}\{\frac{s}{s^2+16}\} = \cos 4t$, $L^{-1}\{\frac{1}{s^2+16}\} = \frac{1}{4}\sin 4t$

$\Rightarrow \boxed{y = \cos 4t - \frac{1}{4}\sin 4t + \frac{1}{4}u_3(t)\sin 4(t-3)}$

8. $Y = \sum_{n=1}^{\infty} y_n$ where $y_n = A_n \cos\frac{n\pi}{2}x + B_n \sin\frac{n\pi}{2}x$

and y_n satisfies

$$y_n'' + 5y_n = \frac{1}{n^2}\cos\frac{n\pi}{2}x - \frac{1}{n^4}\sin\frac{n\pi}{2}x$$

$$y_n'' = -\frac{n^2\pi^2}{4}A_n\cos\frac{n\pi}{2}x - \frac{n^2\pi^2}{4}B_n\sin\frac{n\pi}{2}x$$

$$\Rightarrow \quad y_n'' + 5y_n = \left(5-\frac{n^2\pi^2}{4}\right)A_n\cos\frac{n\pi}{2}x + \left(5-\frac{n^2\pi^2}{4}\right)B_n\sin\frac{n\pi}{2}x$$

$$\Rightarrow \quad \left(5-\frac{n^2\pi^2}{4}\right)A_n = \frac{1}{n^2}, \quad \left(5-\frac{n^2\pi^2}{4}\right)B_n = -\frac{1}{n^4}$$

$$\Rightarrow \quad A_n = \frac{1}{n^2\left(5-\frac{n^2\pi^2}{4}\right)}, \quad B_n = \frac{-1}{n^4\left(5-\frac{n^2\pi^2}{4}\right)}$$

$$\Rightarrow \boxed{Y = \sum_{n=1}^{\infty}\left(\frac{\cos\frac{n\pi}{2}x}{n^2\left(5-\frac{n^2\pi^2}{4}\right)} - \frac{\sin\frac{n\pi}{2}x}{n^4\left(5-\frac{n^2\pi^2}{4}\right)}\right)}$$

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9. $p(\lambda) = \det\begin{bmatrix}1-\lambda & -2\\ 2 & 1-\lambda\end{bmatrix} = (1-\lambda)^2+4 = 0 \Rightarrow \lambda = 1\pm 2i.$

Eigenvalues are $1\pm 2i$. Find the eigenvectors associated to $\lambda = 1+2i \Rightarrow A-(1+2i)I = \begin{bmatrix}-2i & -2\\ 2 & -2i\end{bmatrix} \xrightarrow[\text{reduce}]{\text{Row}} \begin{bmatrix}1 & -i\\ 0 & 0\end{bmatrix}$

$\Rightarrow$ eigenvector $= c\begin{bmatrix}i\\1\end{bmatrix}$ $c\neq 0$. A complex solution to $x' = Ax$ is

$$x(t) = e^{(1+2i)t}\begin{bmatrix}i\\1\end{bmatrix} = e^t(\cos 2t + i\sin 2t)\begin{bmatrix}i\\1\end{bmatrix}$$

$$= e^t\begin{bmatrix}-\sin 2t + i\cos 2t\\ \cos 2t + i\sin 2t\end{bmatrix}. \Rightarrow \text{realvalued solutions are}$$

$$x(t) = c_1 e^t\begin{bmatrix}-\sin 2t\\ \cos 2t\end{bmatrix} + c_2 e^t\begin{bmatrix}\cos 2t\\ \sin 2t\end{bmatrix}$$

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10. $p(\lambda) = \det\begin{bmatrix}1-\lambda & 0\\ 5 & 2-\lambda\end{bmatrix} = (1-\lambda)(2-\lambda) \Rightarrow$ eigenvalues are 1, 2.

Find eigenvectors:

$\lambda = 1$: $A-\lambda I = A - I = \begin{bmatrix}0 & 0\\ 5 & 1\end{bmatrix} \Rightarrow v = \begin{bmatrix}1\\-5\end{bmatrix}.$

$$\lambda = 2 \Rightarrow A - \lambda I = A - 2I = \begin{bmatrix} -1 & 0 \\ 5 & 0 \end{bmatrix} \Rightarrow v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Thus $x(t) = c_1 e^{t} \begin{bmatrix} 1 \\ -5 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 e^{t} \\ -5c_1 e^{t} + c_2 e^{2t} \end{bmatrix}$

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11. (a) $\dfrac{s-3}{s^2+10s+9} = \dfrac{s-3}{(s+9)(s+1)} = \dfrac{A}{s+9} + \dfrac{B}{s+1}$

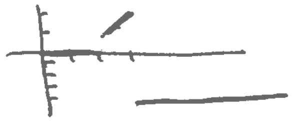
$\Rightarrow A(s+1) + B(s+9) = s-3$ $\quad s=-1 \Rightarrow 8B = -4 \Rightarrow B = -\frac{1}{2}.$

$s = -9 \Rightarrow -8A = -12 \Rightarrow A = 3/2 \Rightarrow \dfrac{s-3}{s^2+10s+9} = \dfrac{3}{2} \cdot \dfrac{1}{s+9} - \dfrac{1}{2} \dfrac{1}{s+1}$

$\Rightarrow L^{-1}\left(\dfrac{s-3}{s^2+10s+9}\right) = \dfrac{3}{2} e^{-9t} - \dfrac{1}{2} e^{-t}$

$\Rightarrow L^{-1}\left(\dfrac{s-3}{s^2+10s+9} \cdot e^{-2s}\right) = u_2(t)\left(\dfrac{3}{2} e^{-9(t-2)} - \dfrac{1}{2} e^{-(t-2)}\right)$

(b)



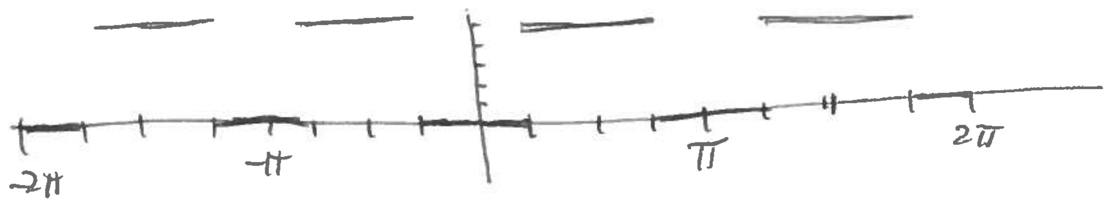
$f(t) = (t-1)\,\chi_{[2,3]}(t) - 4u_3(t)$

$= (t-1)(u_2(t) - u_3(t)) - 4u_3(t) = (t-2)u_2(t) + u_2(t) - (t-3)u_3(t) - 6u_3(t)$

$\Rightarrow F(s) = \dfrac{e^{-2s}}{s^2} + \dfrac{e^{-2s}}{s} - \dfrac{e^{-3s}}{s^2} - \dfrac{6e^{-3s}}{s}$

12.

(a)



(b) $f(x)$ is even so we need only comput a_n. $T = \pi$ so $c = \frac{\pi}{2}$; $a_0 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)\,dx = \dfrac{2}{\pi}\int_{\pi/4}^{3\pi/4} 5\,dx = \dfrac{10}{\pi} \cdot \dfrac{\pi}{2} = 5$

$a_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)\cos 2nx\,dx = \dfrac{2}{\pi}\int_{\pi/4}^{3\pi/4} 5\cos 2nx\,dx = \dfrac{10}{\pi} \cdot \dfrac{1}{2n} \sin 2nx \Big|_{\pi/4}^{3\pi/4}$

$= \dfrac{5}{\pi n}\left[\sin \dfrac{6n\pi}{4} - \sin \dfrac{2n\pi}{4}\right] = \dfrac{5}{\pi n}\begin{cases} 0 & n = 2k \\ -2 & n = 2k+1 \end{cases}$

$$\Rightarrow \boxed{f(x) \sim \frac{5}{2} - \frac{10}{\pi} \sum_{k=1}^{\infty} \frac{\cos 2(2k+1)x}{2k+1}}$$

13. $p(\lambda) = \det \begin{bmatrix} 3-\lambda & 2 & 2 \\ 1 & 2-\lambda & 2 \\ -1 & -1 & -\lambda \end{bmatrix} = (3-\lambda)\left[(2-\lambda)(-\lambda)+2\right] - 2\left[-\lambda+2\right] + 2\left[-1 + (2-\lambda)\right]$

$= (3-\lambda)(\lambda^2-2\lambda+2) + 2\lambda - 4 + 2 - 2\lambda$

$= -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = -(\lambda^3 - 5\lambda^2 + 8\lambda - 4) = -(\lambda-1)(\lambda-2)^2$

Eigenvalues are 1, 2 (mult 2).

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14. $A - \lambda I = A + 2I = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 \mapsto R_3 - R_1} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$.

$\to \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow V = c \begin{bmatrix} 1/4 \\ -1/2 \\ 1 \end{bmatrix}$ $\quad c \neq 0$.

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15. (a) $2 = k \cdot 1 \Rightarrow k = 2$ (k = spring constant).

$\Rightarrow \frac{3.2}{32} y'' + .4y' + 2y = 0 \Rightarrow \boxed{y'' + 4y' + 20y = 0}$

(b) $y(0) = -1$, $y'(0) = 0$.

$p(\lambda) = \lambda^2 + 4\lambda + 20 = (\lambda+2)^2 + 16 \Rightarrow \lambda = -2 \pm 4i$.

$\Rightarrow y = c_1 e^{-2t} \cos 4t + c_2 e^{-2t} \sin 4t$

$-1 = y(0) = c_1$, $y' = -2c_1 e^{-2t} \cos 4t - 4c_1 e^{-2t} \sin 4t - 2c_2 e^{-2t} \sin 4t + 4c_2 e^{-2t} \cos 4t$

$0 = y'(0) = -2c_1 + 4c_2 \Rightarrow c_2 = \frac{1}{2} c_1 \Rightarrow c_2 = -\frac{1}{2}$.

$\Rightarrow \boxed{y = -e^{-2t} \cos 4t - \frac{1}{2} e^{-2t} \sin 4t}$

(c) $y = A e^{-2t} \sin(4t + \varphi)$

where $A = \sqrt{(-1)^2 + (\frac{1}{2})^2} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$, $\tan\varphi = 2$

& φ in 4th quadrant $\Rightarrow \varphi = \tan^{-1}(2) + \pi$

$\Rightarrow \varphi = 4.249$.

$\Rightarrow y = \frac{\sqrt{5}}{2} e^{-2t} \sin(4t + 4.249)$.