**Instructions.** Answer each of the questions on your own paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* Put your name on each page of your paper.

1. [**16 Points**] Find all solutions of the following system of linear equations. Be sure to show all your steps!

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & 2x_3 & = & -8 \\ 3x_1 & + & x_2 & + & 9x_3 & = & 1 \\ x_1 & & & + & 4x_3 & = & 2 \end{array}$$

▶ **Solution.** The augmented matrix of the system is

$$\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 & -8 \\ 3 & 1 & 9 & 1 \\ 1 & 0 & 4 & 2 \end{bmatrix}.$$

Now use Gauss-Jordan elimination to row reduce the augmented matrix:

$$\begin{bmatrix} 1 & 2 & -2 & -8 \\ 3 & 1 & 9 & 1 \\ 1 & 0 & 4 & 2 \end{bmatrix} \xrightarrow[t_{13}(-1)]{t_{12}(-3)} \begin{bmatrix} 1 & 2 & -2 & -8 \\ 0 & -5 & 15 & 25 \\ 0 & -2 & 6 & 10 \end{bmatrix}$$

$$\xrightarrow{m_2(-1/5)} \begin{bmatrix} 1 & 2 & -2 & -8 \\ 0 & 1 & -3 & -5 \\ 0 & -2 & 6 & 10 \end{bmatrix} \xrightarrow{t_{23}(2)} \begin{bmatrix} 1 & 2 & -2 & -8 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{t_{21}(-2)} \begin{bmatrix} 1 & 0 & 4 & 2 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The last matrix is in reduced row echelon form. It is the augmented matrix of the linear system

$$\begin{array}{rcrcrcr} x_1 & + & & & 4x_3 & = & 2 \\ & & x_2 & - & 3x_3 & = & -5 \end{array}$$

where the last equations $0 = 0$ is left off. The solutions of this last system are obtained by identifying the bound variables as $x_1$ and $x_2$, so $x_3$ is a free variable. Solving for $x_1$ and $x_2$ in terms of $x_3$ gives:

$$\begin{aligned} x_1 &= 2 - 4x_3 \\ x_2 &= -5 + 3x_3. \end{aligned}$$

Letting the free variable be assigned arbitrarily: $x_3 = \alpha$,gives all the solutions as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 - 4\alpha \\ -5 + 3\alpha \\ \alpha \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 0 \end{bmatrix} + \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix} \alpha$$

where $\alpha$ is an arbitrary real number. ◀

2. [**16 Points**] Let $A = \begin{bmatrix} x-2 & 4 & 3 \\ 1 & x+1 & -2 \\ 0 & 0 & x-4 \end{bmatrix}$.

   (a) Compute $\det A$.

   ▶ **Solution.** Use cofactor expansion along the third row:

$$\det A = (x-4)\det\begin{bmatrix} x-2 & 4 \\ 1 & x+1 \end{bmatrix} = (x-4)((x-2)(x+1)-4)$$
$$= (x-4)(x^2-x-6) = (x-4)(x-3)(x+2).$$

◀

   (b) For which values of $x$ is the matrix *not* invertible.

   ▶ **Solution.** $A$ is not invertible if and only if $\det A = 0$. This happens if and only if $x = 4$, 3, or $-2$. ◀

3. [**16 Points**] Solve the initial value problem: $y' - 4y = 6e^{4t} + 3, \quad y(0) = -2$.

▶ **Solution.** This equation is linear with coefficient function $p(t) = -4$ so that an integrating factor is given by $\mu(t) = e^{\int -4\,dt} = e^{-4t}$. Multiplication of the differential equation by the integrating factor gives

$$e^{-4t}y' - 4e^{-4t}y = 6 + 3e^{-4t},$$

and the left hand side is recognized (by the choice of $\mu(t)$) as a perfect derivative:

$$\frac{d}{dt}(e^{-4t}y) = 6 + 3e^{-4t}.$$

Integration then gives

$$e^{-4t}y = 6t - \frac{3}{4}e^{-4t} + C,$$

where $C$ is an integration constant. Multiplying through by $e^{4t}$ gives

$$y = 6te^{4t} - \frac{3}{4} + Ce^{4t}.$$

The initial condition gives $-2 = y(0) = -\frac{3}{4} + C$ so $C = -\frac{5}{4}$ and thus

$$\boxed{y = 6te^{4t} - \frac{3}{4} - \frac{5}{4}e^{4t}.}$$

◀

4. [**16 Points**] Solve the initial value problem: $ty' - 3y = t^5 \quad y(1) = 3$.

▶ **Solution.** This equation is linear. Putting it in standard form gives

$$y' - \frac{3}{t}y = t^4.$$

Thus, the coefficient function is $p(t) = -\frac{3}{t}$, so that an integrating factor is $\mu(t) = e^{\int -\frac{3}{t}\,dt} = e^{-3\ln t} = e^{\ln(t^{-3})} = t^{-3}$. Multiplication of the differential equation (in standard form) by the integrating factor gives

$$t^{-3}y' - 3t^{-4}y = t,$$

and the left hand side is recognized (by the choice of $\mu(t)$) as a perfect derivative:

$$\frac{d}{dt}(t^{-3}y) = t.$$

Integration then gives

$$t^{-3}y = \frac{t^2}{2} + C,$$

where $C$ is an integration constant. Multiplying through by $t^3$ gives

$$y = \frac{t^5}{2} + Ct^3.$$

The initial condition $y(1) = 3$ implies $3 = y(1) = \frac{1}{2} + C$ so that $C = 5/2$. Hence

$$\boxed{y = \frac{t^5}{2} + \frac{5}{2}t^3.}$$

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5. **[16 Points]** Solve the initial value problem: $y' = y^2(1+t), \quad y(0) = -2$.

▶ **Solution.** This equation is separable. There is one equilibrium solution: $y = 0$. This does not satisfy the initial condition so it is not the required solution. If $y \neq 0$, separate variables to get $y^{-2}y' = 1 + t$, which in differential form is

$$y^{-2}\,dy = (1+t)\,dt$$

and integration then gives $-y^{-1} = (1+t)^2/2 + C$. The initial condition $y(0) = -2$ gives $-(-2)^{-1} = 1/2 + C$ so $C = 0$, and $-y^{-1} = (1+t)^2/2$. Solving for $y$ gives

$$\boxed{y = \frac{-2}{(1+t)^2}.}$$

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6. **[20 Points]** A tank initially contains 300 gallons of a salt solution made by dissolving 30 pounds of salt in water. A solution containing 0.6 pounds of salt per gallon enters the tank at a rate of 5 gallons per minute. A drain is opened at the bottom of the tank through which the well stirred solution leaves the tank at the same rate of 5 gallons per minute. Let $y(t)$ denote the amount of salt (in pounds) which is in the tank at time $t$.

   (a) What is $y(0)$? $\boxed{y(0) = 30}$

   (b) Write the differential equation that $y(t)$ must satisfy.

   ▶ **Solution.** The balance equation is

$$y'(t) = \text{rate in} - \text{rate out}.$$

   The rate in is 0.6 lb/gal × 5 gal/min, i.e., 3 lb/min. The rate out is

$$\left(\frac{y(t)}{V(t)}\right) \times 3.$$

   Since mixture is entering and leaving at the same volume rate of 5 gal/min, the volume of mixture in the tank is constant. Thus $V(t) = 300$. Hence $y(t)$ satisfies the equation

$$y' = 3 - \frac{5}{300}y,$$

   so that the initial value problem satisfied by $y(t)$ is

$$\boxed{y' + \frac{1}{60}y = 3, \qquad y(0) = 30.}$$

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   (c) Solve the differential equation to find $y(t)$.

   ▶ **Solution.** This is a linear differential equation with integrating factor $\mu(t) = e^{t/60}$, so multiplication of the differential equation by $\mu(t)$ gives an equation

$$\frac{d}{dt}\left(e^{t/60}y\right) = 3e^{t/60}.$$

   Integration of this equation gives

$$e^{t/60}y = 180e^{t/60} + C,$$

   where $C$ is an integration constant. Dividing by $e^{t/60}$ gives

$$y = 180 + Ce^{-t/60},$$

   and the initial condition $y(0) = 30$ gives a value of $C = -150$. Hence, the amount of salt at time $t$ is

$$\boxed{y(t) = 180 - 150e^{-t/60}.}$$

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(d) How much salt is in the tank after 1/2 hour?

▶ **Solution.** This is obtained by taking $t = 30$ (minutes) in the previous equation:

$$\boxed{y(30) = 180 - 150e^{-30/60} \approx 89.02\text{lb}}$$

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