Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* A table of Laplace transforms, a table of convolutions, and the statement of the main partial fraction decomposition theorem have been appended to the exam.

1. [**24 Points**] Compute the Laplace transform of each of the following functions. You may use the attached tables, but be sure to identify which formulas you are using by citing the number(s) or name of the formula in the table.

   (a) $f_1(t) = e^{-2t}(t^3 - \cos 3t)$

   ▶ **Solution.** $f_1(t) = e^{-2t}t^3 - e^{-2t}\cos 3t)$ so

$$F_1(s) = \left(\frac{6}{s^4} - \frac{s}{s^2+3^2}\right)\Bigg|_{s\mapsto s-(-2)} = \frac{6}{(s+2)^4} - \frac{s+2}{(s+2)^2+9}.$$

◀

   (b) $f_2(t) = (t + e^{-3t})^2$

   ▶ **Solution.** Multiply out to get $f_2(t) = t^2 + 2te^{-3t} + e^{-6t}$. Thus

$$F_2(s) = \frac{2}{s^3} + \frac{2}{(s+3)^2} + \frac{1}{s+6}.$$

◀

   (c) $f_3(t) = \cos 4t * \cos 4t$. Recall that $f * g$ is the *convolution* product of f and g.

   ▶ **Solution.** From the convolution principle,

$$F_3(s) = \left(\mathcal{L}\left\{\cos 4t\right\}\right)^2 = \left(\frac{s}{s^2+16}\right)^2 = \frac{s^2}{(s^2+16)^2}.$$

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2. [**9 Points**] Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial value problem.

$$2y'' + 5y' + 2y = \cos 2t, \qquad y(0) = 3,\ y'(0) = -2.$$

   Note that you are asked to find $Y(s)$, but *not* $y(t)$.

▶ **Solution.** Apply the Laplace transform to the differential equation, using linearity and the input derivative principle to get

$$2(s^2Y(s) - sy(0) - y'(0)) + 5(sY(s) - y(0)) + 2Y(s) = \frac{s}{s^2+4}.$$

This gives

$$s^2Y(s) - 6s + 4 + 5Y(s) - 15 + 2Y(s) = \frac{s}{s^2+4}$$

or

$$(s^2 + 5s + 2)Y(s) - 6s - 11 = \frac{s}{s^2+4}.$$

Solve for $Y(s)$ to get

$$Y(s) = \frac{6s+11}{2s^2+5s+2} + \frac{s}{(2s^2+5s+2)(s^2+4)}.$$

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3. [**16 Points**] Compute the inverse Laplace transform of each of the following rational functions.

   (a) $F(s) = \dfrac{s^2+s+6}{(s+1)^2(s-1)}$

   ▶ **Solution.** Expand $F(s)$ in partial fractions, starting with the $s-1$ term. Thus,

$$F(s) = \frac{A}{s-1} + \frac{p_1(s)}{(s+1)^2}$$

   where

$$A = \left.\frac{s^2+s+6}{(s+1)^2}\right|_{s=1} = \frac{8}{4} = 2,$$

   and

$$\begin{aligned} p_1(s) &= \frac{(s^2+s+6) - 2(s+1)^2}{s-1} \\ &= \frac{s^2+s+6-2(s^2+2s+1)}{s-1} \\ &= \frac{-s^2-3s+4}{s-1} = \frac{-(s^2+3s-4)}{s-1} \\ &= \frac{-(s-1)(s+4)}{s-1} = -(s+4). \end{aligned}$$

   Since

$$\frac{-(s+4)}{(s+1)^2} = -\frac{(s+1)+3}{(s+1)^2} = -\frac{1}{s+1)^2} - \frac{3}{(s+1)^2}$$

we get

$$\frac{s^2+s+6}{(s+1)^2(s-1)} = \frac{2}{s-1} - \frac{1}{s+1} - \frac{3}{(s+1)^2}.$$

Then

$$\mathcal{L}^{-1}\left\{\frac{s^2+s+6}{(s+1)^2(s-1)}\right\} = 2e^t - e^{-t} - 3te^{-t}.$$

◀

(b) $G(s) = \dfrac{2s+5}{s^2+4s+29}$

▶ **Solution.**

$$\begin{aligned} G(s) = \frac{2s+5}{s^2+4s+29} &= \frac{2s+5}{(s+2)^2+25} \\ &= \frac{2((s+2)-2)+5}{(s+2)^2+25} = \frac{2(s+2)+1}{(s+2)^2+25} \\ &= \frac{2(s+2)}{(s+2)^2+25} + \frac{1}{(s+2)^2+25}. \end{aligned}$$

Thus

$$\mathcal{L}^{-1}\left\{G(s)\right\} = 2e^{-2t}\cos 5t + \frac{1}{5}e^{-2t}\sin 5t.$$

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4. [**36 Points**] Find the general solution of each of the following homogeneous differential equations.

(a) $y'' + 4y = 0$

▶ **Solution.** $q(s) = s^2+4$ which has roots $\pm 2i$. Thus, $y = c_1\cos 2t + c_2\sin 2t$. ◀

(b) $y'' + 4y' + 3y = 0$

▶ **Solution.** $q(s) = s^2+4s+3 = (s+3)(s+1)$ which has roots -3 and -1. Thus,

$$y = c_1e^{-3t} + c_2e^{-t}.$$

◀

(c) $y'' + 4y' + 4y = 0$

▶ **Solution.** $q(s) = s^2+4s+4 = (s+2)^2$ which has a single root -2 with multiplicity 2. Thus,

$$y = c + 1e^{-2t} + c_2te^{-2t}.$$

◀

(d) $y'' + 4y' + 13y = 0$

► **Solution.** $q(s) = s^2 + 4s + 13 = (s+2)^2 + 9$ so the roots are $-2 \pm 3i$. Hence,

$$y = c_1 e^{-2t}\cos 3t + c_2 e^{-2t}\sin 3t.$$

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5. [**15 Points**] Find the general solution of the following differential equation:

$$y'' + 6y' + 8y = 5e^{-4t}.$$

You may use whatever method you prefer.

► **Solution.** Use the method of undetermined coefficients. The characteristic polynomial is $q(s) = s^2 + 6s + 8 = (s+2)(s+4)$ which has roots -2 and -4. Thus $\mathcal{B}_q = \{e^{-2t}, e^{-4t}\}$ and $y_h = c_1 e^{-2t} + c_2 e^{-4t}$. Since $\mathcal{L}\left\{5e^{-4t}\right\} = \dfrac{5}{s+4}$ the denominator is $v = s + 4$ and $qv = (s+2)(s+4)^2$. Hence,

$$\mathcal{B}_{qv} \setminus \mathcal{B}_q = \left\{e^{-2t}, e^{-4t}, te^{-4t}\right\} \setminus \left\{e^{-2t}, e^{-4t}\right\} = \left\{te^{-4t}\right\}.$$

Therefore, the test function for y_p is $y_p = Ate^{-4t}$. Compute the derivatives:

$$\begin{aligned} y_p' &= A(1-4t)e^{-4t} \\ y_p'' &= A(-8+16t)e^{-4t}. \end{aligned}$$

Substituting into the differential equation gives

$$\begin{aligned} y_p'' + 6y_p' + 8y_p &= A(-8+16t)e^{-4t} + 6(a(1-4t)e^{-4t} + 8Ate^{-4t} \\ &= (-8+6)Ae^{-4t} = 5e^{-4t}. \end{aligned}$$

Thus, $-2A = 5$ so $A = -5/2$ and $y_p = (-5/2)te^{-4t}$ and the general solution is

$$y_g = y_h + y_p = c_1 e^{-2t} + c_2 e^{-4t} - \frac{5}{2}te^{-4t}.$$

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Laplace Transform Table

	$f(t)$	$\to$	$F(s) = \mathcal{L}\left\{f(t)\right\}(s)$
1.	1	$\to$	$\frac{1}{s}$
2.	t^n	$\to$	$\frac{n!}{s^{n+1}}$
3.	e^{at}	$\to$	$\frac{1}{s-a}$
4.	$t^n e^{at}$	$\to$	$\frac{n!}{(s-a)^{n+1}}$
5.	$\cos bt$	$\to$	$\frac{s}{s^2+b^2}$
6.	$\sin bt$	$\to$	$\frac{b}{s^2+b^2}$
7.	$e^{at}\cos bt$	$\to$	$\frac{s-a}{(s-a)^2+b^2}$
8.	$e^{at}\sin bt$	$\to$	$\frac{b}{(s-a)^2+b^2}$
9.	$\frac{1}{2b^2}(\sin bt - bt\cos bt)$	$\to$	$\frac{b}{(s^2+b^2)^2}$
10.	$\frac{1}{2b}t\sin bt$	$\to$	$\frac{s}{(s^2+b^2)^2}$

Laplace Transform Principles

Linearity	$\mathcal{L}\left\{af(t)+bg(t)\right\}$	$=$	$a\mathcal{L}\left\{f\right\}+b\mathcal{L}\left\{g\right\}$
Input Derivative Principles	$\mathcal{L}\left\{f'(t)\right\}(s)$	$=$	$s\mathcal{L}\left\{f(t)\right\}-f(0)$
	$\mathcal{L}\left\{f''(t)\right\}(s)$	$=$	$s^2\mathcal{L}\left\{f(t)\right\}-sf(0)-f'(0)$
First Translation Principle	$\mathcal{L}\left\{e^{at}f(t)\right\}$	$=$	$F(s-a)$
Transform Derivative Principle	$\mathcal{L}\left\{-tf(t)\right\}(s)$	$=$	$\frac{d}{ds}F(s)$
The Dilation Principle	$\mathcal{L}\left\{f(bt)\right\}(s)$	$=$	$\frac{1}{b}\mathcal{L}\left\{f(t)\right\}(s/b)$
The Convolution Principle	$\mathcal{L}\left\{(f*g)(t)\right\}(s)$	$=$	$F(s)G(s).$

Table of Convolutions

	$f(t)$	$g(t)$	$(f * g)(t)$
1.	1	$g(t)$	$\int_0^t g(\tau)\,d\tau$
2.	t^m	t^n	$\frac{m!n!}{(m+n+1)!}t^{m+n+1}$
3.	t	$\sin at$	$\frac{at - \sin at}{a^2}$
4.	t^2	$\sin at$	$\frac{2}{a^3}(\cos at - (1 - \frac{a^2t^2}{2}))$
5.	t	$\cos at$	$\frac{1 - \cos at}{a^2}$
6.	t^2	$\cos at$	$\frac{2}{a^3}(at - \sin at)$
7.	t	e^{at}	$\frac{e^{at} - (1 + at)}{a^2}$
8.	t^2	e^{at}	$\frac{2}{a^3}(e^{at} - (a + at + \frac{a^2t^2}{2}))$
9.	e^{at}	e^{bt}	$\frac{1}{b-a}(e^{bt} - e^{at}) \quad a \neq b$
10.	e^{at}	e^{at}	te^{at}
11.	e^{at}	$\sin bt$	$\frac{1}{a^2+b^2}(be^{at} - b\cos bt - a\sin bt)$
12.	e^{at}	$\cos bt$	$\frac{1}{a^2+b^2}(ae^{at} - a\cos bt + b\sin bt)$
13.	$\sin at$	$\sin bt$	$\frac{1}{b^2-a^2}(b\sin at - a\sin bt) \quad a \neq b$
14.	$\sin at$	$\sin at$	$\frac{1}{2a}(\sin at - at\cos at)$
15.	$\sin at$	$\cos bt$	$\frac{1}{b^2-a^2}(a\cos at - a\cos bt) \quad a \neq b$
16.	$\sin at$	$\cos at$	$\frac{1}{2}t\sin at$
17.	$\cos at$	$\cos bt$	$\frac{1}{a^2-b^2}(a\sin at - b\sin bt) \quad a \neq b$
18.	$\cos at$	$\cos at$	$\frac{1}{2a}(at\cos at + \sin at)$

Partial Fraction Expansion Theorems

The following two theorems are the main partial fractions expansion theorems, as presented in the text.

Theorem 1 (Linear Case). *Suppose a proper rational function can be written in the form*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)}$$

and $q(\lambda) \neq 0$. *Then there is a unique number* A_1 *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)} = \frac{A_1}{(s-\lambda)^n} + \frac{p_1(s)}{(s-\lambda)^{n-1} q(s)}. \tag{1}$$

The number A_1 *and the polynomial* $p_1(s)$ *are given by*

$$A_1 = \frac{p_0(\lambda)}{q(\lambda)} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - A_1 q(s)}{s-\lambda}. \tag{2}$$

Theorem 2 (Irreducible Quadratic Case). *Suppose a real proper rational function can be written in the form*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)},$$

where s^2+cs+d *is an irreducible quadratic that is factored completely out of* $q(s)$. *Then there is a unique linear term* $B_1 s + C_1$ *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)} = \frac{B_1 s + C_1}{(s^2+cs+d)^n} + \frac{p_1(s)}{(s^s+cs+d)^{n-1} q(s)}. \tag{3}$$

If $a+ib$ *is a complex root of* s^2+cs+d *then* $B_1 s + C_1$ *and the polynomial* $p_1(s)$ *are given by*

$$B_1(a+ib) + C_1 = \frac{p_0(a+ib)}{q(a+ib)} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - (B_1 s + C_1) q(s)}{s^2+cs+d}. \tag{4}$$