Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* A table of Laplace transforms has been appended to the exam. The following trigonometric identities may also be of use:

$$\boxed{\begin{aligned} \sin(\theta+\varphi) &= \sin\theta\cos\varphi + \sin\varphi\cos\theta \\ \cos(\theta+\varphi) &= \cos\theta\cos\varphi - \sin\theta\sin\varphi \end{aligned}}$$

1. **[18 Points]** Find the the general solution of the following Cauchy-Euler equations:

   (a) $2t^2y'' - 3ty' + 2y = 0$.

   ▶ **Solution.** The indicial polynomial is

   $$Q(s) = 2s(s-1) - 3s + 2 = 2s^2 - 5s + 2 = (2s-1)(s-2),$$

   which has the two distinct real roots 1/2 and 2. Hence the general solution is

   $$\boxed{y = c_1t^{1/2} + c_2t^2.}$$

   ◀

   (b) $t^2y'' + ty' + 16y = 0$.

   ▶ **Solution.** The indicial polynomial is

   $$Q(s) = s(s-1) + s + 16 = s^2 + 16,$$

   which has the complex roots $-4i$. Hence the general solution is

   $$\boxed{y = c_1\cos(4\ln|t|) + c_2\sin(4\ln|t|).}$$

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2. **[18 Points]** Use *variation of parameters* to find a particular solution of the nonhomogeneous differential equation

   $$t^2y'' - 4ty' + 6y = t^3.$$

   You may assume that the solution of the homogeneous equation $t^2y'' - 4ty' + 6y = 0$ is $y_h = c_1t^2 + c_2t^3$.

   ▶ **Solution.** Letting $y_1 = t^2$ and $y_2 = t^3$, a particular solution has the form

   $$y_p = u_1y_1 + u_2y_2 = u_1t^2 + u_2t^3,$$

where u_1 and u_2 are unknown functions whose derivatives satisfy the simultaneous equations

$$\begin{aligned} u_1' t^2 + u_2' t^3 &= 0 \\ 2u_1' t + 3u_2' t^2 &= t. \end{aligned}$$

Applying Cramer's rule gives

$$u_1' = \frac{\begin{vmatrix} 0 & t \\ t^3 & 3t^2 \end{vmatrix}}{\begin{vmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{vmatrix}} = \frac{-t^4}{t^4} = -1$$

and

$$u_2' = \frac{\begin{vmatrix} t^2 & 0 \\ 2t & t \end{vmatrix}}{\begin{vmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{vmatrix}} = \frac{t^3}{t^4} = t^{-1}$$

Integrating gives $u_1 = -t$ and $u_2 = \ln|t|$ so that

$$y_p = (-t)t^2 + (\ln|t|)t^3.$$

Hence

$$\boxed{y_p = -t^3 + (\ln|t|)t^3.}$$

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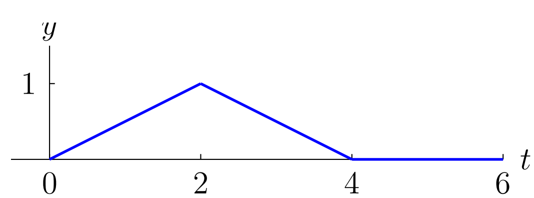
3. **[18 Points]** Let f be the function defined by

$$f(t) = \begin{cases} \frac{t}{2} & \text{if } 0 \le t < 2, \\ \frac{t}{2} - 1 & \text{if } 2 \le t < 4, \\ 0 & \text{if } t \ge 4. \end{cases}$$

(a) Sketch the graph of $f(t)$ over the interval [0, 6].



Graph of $f(t)$

(b) Find the Laplace transform of $f(t)$.

▶ **Solution.** Use characteristic functions to write $f(t)$ in terms of unit step functions:

$$\begin{aligned} f(t) &= \frac{t}{2}\chi_{[0,\,2)}(t) + \big(\frac{t}{2} - 1\big)\chi_{[2,\,4)}(t) \\ &= \frac{t}{2}(h(t) - h(t-2)) + \big(\frac{t}{2} - 1\big)(h(t-2) - h(t-4)) \\ &= \frac{t}{2} - h(t-2) + \big(1 - \frac{t}{2}\big)h(t-4). \end{aligned}$$

Now apply the second translation theorem to get

$$\begin{aligned} F(s) = \mathcal{L}\left\{f(t)\right\} &= \frac{1}{2s^2} + e^{-2s}\frac{1}{s} + e^{-4s}\mathcal{L}\left\{1 - \frac{t+4}{2}\right\} \\ &= \frac{1}{2s^2} + e^{-2s}\frac{1}{s} + e^{-4s}\left(-\frac{1}{s} - \frac{1}{2s^2}\right) \end{aligned}$$

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4. [**18 Points**]

(a) Find the inverse Laplace transform $f(t)$ of the following function:

$$F(s) = \frac{1 + e^{-\pi s}}{s^2 + 1}$$

▶ **Solution.**

$$\begin{aligned} f(t) &= \sin t + h(t-\pi)\sin(t-\pi) = \sin t - h(t-\pi)\sin t \\ &= \begin{cases} \sin t & \text{if } 0 \le t < \pi \\ 0 & \text{if } t \ge \pi \end{cases} \end{aligned}$$

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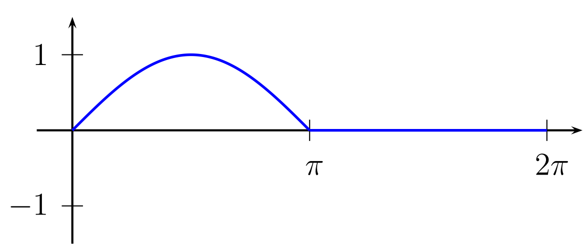
(b) Sketch the graph of $f(t)$.



5. [**18 Points**] Solve the following initial value problem:

$$y'' + y = \delta_0(t) - \delta_\pi(t) + \delta_{2\pi}(t), \qquad y(0) = 0,\ y'(0) = 0.$$

(Remember that $\delta_c(t)$ is the Dirac delta function.)

Give a careful sketch of the graph of the solution for the interval $0 \le t \le 3\pi$.

▶ **Solution.** Let $Y(s) = \mathcal{L}\{y(t)\}$ be the Laplace transform of the solution function. Apply the Laplace transform to both sides of the equation to get

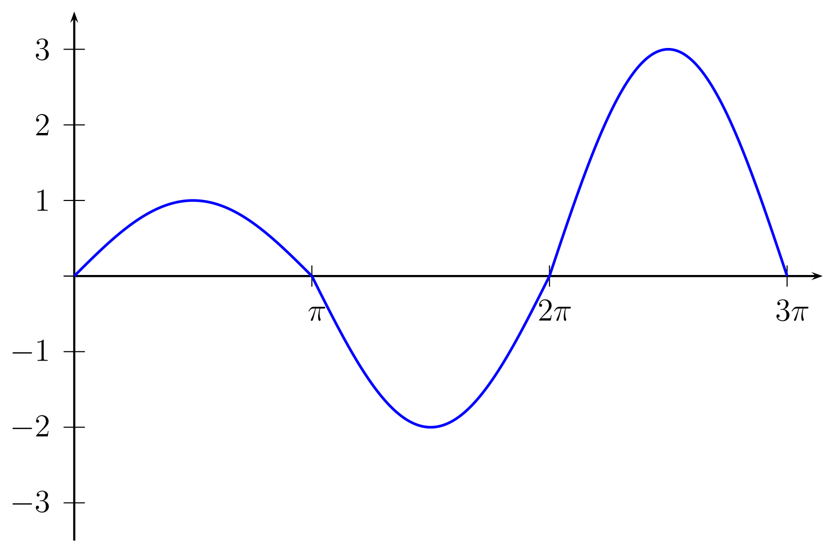
$$(s^2+1)Y(s) = 1 - e^{-\pi s} + e^{-2\pi s}.$$

Thus,

$$Y(s) = \frac{1}{s^2+1} - e^{-\pi s}\frac{1}{s^2+1} + e^{-2\pi s}\frac{1}{s^2+1}.$$

Apply the inverse Laplace transform to get

$$\begin{aligned} y(t) &= \sin t - h(t-\pi)\sin(t-\pi) + h(t-2\pi)\sin(t-2\pi) \\ &= \begin{cases} \sin t & \text{if } 0 \le t < \pi, \\ 2\sin t & \text{if } \pi \le t < 2\pi, \\ 3\sin t & \text{if } t \ge 2\pi. \end{cases} \end{aligned}$$



Graph of $y(t)$.

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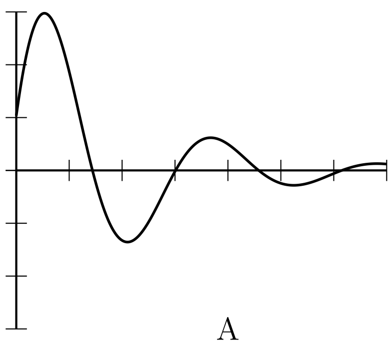
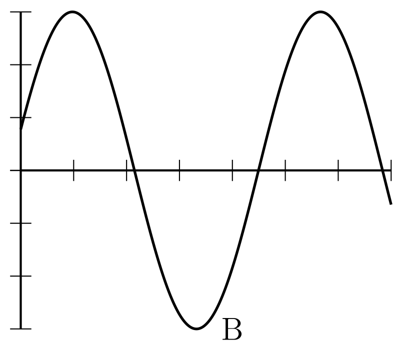
6. [**10 Points**] A mass-spring-dashpot system with forcing function $f(t)$ satisfies the differential equation

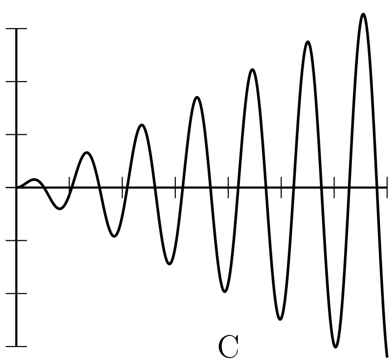
$$my'' + \mu y' + ky = f(t)$$

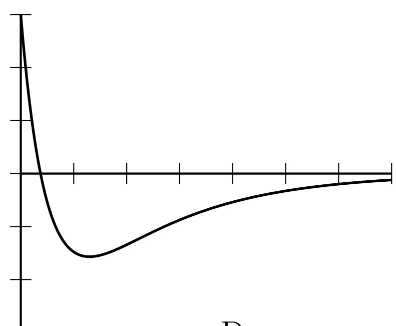
where m is the mass, μ the damping constant, and k is the spring constant. Each of the following differential equations represents such a system. Each of the graphs below is the graph of a solution of (exactly) one of these differential equations. Identify each graph with the equation by placing the appropriate letter in the following table. It may be useful to remember what it means for a spring system to be undamped, underdamped, or overdamped.

Equation	**Graph**
$9y'' + 4y = 0$	**B**
$2y'' + 5y' + 2y = 0$	**D**
$y'' + y' + \frac{5}{4}y = 0$	**A**
$y'' + 9y = \cos 3t$	**C**


A


B


C


D

The equation $9y'' + 4y = 0$ represents undamped harmonic motion, which has a graph $y = A\cos(2t/3 - \delta)$. Graph **B** is the only one with constant amplitude. Graph **A** represents underdamped motion. The equation $y'' + y' + \frac{5}{4}y = 0$ has discriminant $D = -5 < 0$ so it is underdamped and $2y'' + 5y' + 2y = 0$ has discriminant $D = 9 > 0$, which is ovedamped. The only overdamped graph is **D**. The equation $y'' + 9y = \cos 3t$ represents undamped forced motion with the natural frequency and forcing frequency both equal to 3. Thus, the amplitude will increase without bound. This is represented in graph **C**.

Laplace Transform Table

	$f(t)$	$\longleftrightarrow$	$F(s) = \mathcal{L}\left\{f(t)\right\}(s)$
1.	1	$\longleftrightarrow$	$\dfrac{1}{s}$
2.	t^n	$\longleftrightarrow$	$\dfrac{n!}{s^{n+1}}$
3.	e^{at}	$\longleftrightarrow$	$\dfrac{1}{s-a}$
4.	$t^n e^{at}$	$\longleftrightarrow$	$\dfrac{n!}{(s-a)^{n+1}}$
5.	$\cos bt$	$\longleftrightarrow$	$\dfrac{s}{s^2+b^2}$
6.	$\sin bt$	$\longleftrightarrow$	$\dfrac{b}{s^2+b^2}$
7.	$e^{at}\cos bt$	$\longleftrightarrow$	$\dfrac{s-a}{(s-a)^2+b^2}$
8.	$e^{at}\sin bt$	$\longleftrightarrow$	$\dfrac{b}{(s-a)^2+b^2}$
9.	$h(t-c)$	$\longleftrightarrow$	$\dfrac{e^{-sc}}{s}$
10.	$\delta_c(t) = \delta(t-c)$	$\longleftrightarrow$	e^{-sc}

Laplace Transform Principles

Linearity	$\mathcal{L}\left\{af(t)+bg(t)\right\}$	$=$	$a\mathcal{L}\left\{f\right\}+b\mathcal{L}\left\{g\right\}$
Input Derivative Principles	$\mathcal{L}\left\{f'(t)\right\}(s)$	$=$	$s\mathcal{L}\left\{f(t)\right\}-f(0)$
	$\mathcal{L}\left\{f''(t)\right\}(s)$	$=$	$s^2\mathcal{L}\left\{f(t)\right\}-sf(0)-f'(0)$
First Translation Principle	$\mathcal{L}\left\{e^{at}f(t)\right\}$	$=$	$F(s-a)$
Transform Derivative Principle	$\mathcal{L}\left\{-tf(t)\right\}(s)$	$=$	$\dfrac{d}{ds}F(s)$
Second Translation Principle	$\mathcal{L}\left\{h(t-c)f(t-c)\right\}$	$=$	$e^{-sc}F(s)$, or
	$\mathcal{L}\left\{g(t)h(t-c)\right\}$	$=$	$e^{-sc}\mathcal{L}\left\{g(t+c)\right\}$.
The Convolution Principle	$\mathcal{L}\left\{(f*g)(t)\right\}(s)$	$=$	$F(s)G(s)$.

Partial Fraction Expansion Theorems

The following two theorems are the main partial fractions expansion theorems, as presented in the text.

Theorem 1 (Linear Case). *Suppose a proper rational function can be written in the form*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)}$$

and $q(\lambda) \neq 0$. *Then there is a unique number* A_1 *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)} = \frac{A_1}{(s-\lambda)^n} + \frac{p_1(s)}{(s-\lambda)^{n-1} q(s)}. \tag{1}$$

The number A_1 *and the polynomial* $p_1(s)$ *are given by*

$$A_1 = \frac{p_0(\lambda)}{q(\lambda)} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - A_1 q(s)}{s-\lambda}. \tag{2}$$

Theorem 2 (Irreducible Quadratic Case). *Suppose a real proper rational function can be written in the form*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)},$$

where s^2+cs+d *is an irreducible quadratic that is factored completely out of* $q(s)$. *Then there is a unique linear term* $B_1 s + C_1$ *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)} = \frac{B_1 s + C_1}{(s^2+cs+d)^n} + \frac{p_1(s)}{(s^s+cs+d)^{n-1} q(s)}. \tag{3}$$

If $a+ib$ *is a complex root of* s^2+cs+d *then* $B_1 s + C_1$ *and the polynomial* $p_1(s)$ *are given by*

$$B_1(a+ib) + C_1 = \frac{p_0(a+ib)}{q(a+ib)} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - (B_1 s + C_1) q(s)}{s^2+cs+d}. \tag{4}$$