Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* A table of Laplace transforms and the statement of the partial fraction decomposition theorems are attached to the exam.

In Exercises 1 – 6, solve the given differential equation. If initial values are given, solve the initial value problem. Otherwise, give the general solution. Some problems may be solvable by more than one technique. You are free to choose whatever technique that you deem to be most appropriate.

1. [**12 Points**] $y' + 2y = e^{2t} + e^{-2t}, \quad y(0) = -1.$
2. [**9 Points**] $y'' + 4y' + 13y = 0.$
3. [**9 Points**] $4y'' + 20y' + 25y = 0.$
4. [**12 Points**] $3y'' + 11y' + 10y = 0, \quad y(0) = 0,\ y'(0) = 2.$
5. [**12 Points**] $2t^2y'' + 5ty' - 2y = 0, \qquad y(1) = -1,\ y'(1) = 3.$
6. [**12 Points**] $y'' + 2y' + y = 2\sin 3t.$
7. [**12 Points**] Find the general solution for $t > 0$ of the differential equation
$$t^2y'' + 2ty' - 2y = \sqrt{t},$$
given the fact that two solutions of the associated homogeneous equation are $y_1(t) = t$ and $y_2(t) = t^{-2}$.
8. [**16 Points**] Let $f(t)$ be the following function:
$$f(t) = \begin{cases} 1 & \text{if } 0 \le t < 2, \\ t-1 & \text{if } t \ge 2. \end{cases}$$
   (a) Compute the Laplace transform $F(s)$ of $f(t)$.
   (b) Find the Laplace transform $Y(s)$ of the solution of the initial value problem
$$y'' + 2y' + 5y = f(t), \qquad y(0) = 2,\ y'(0) = -1,$$
   DO NOT solve for $y(t)$. Just find $Y(s)$. You may express your answer in terms of $F(s)$.
9. [**16 Points**] Compute the inverse Laplace transform of the following functions:
   (a) $F(s) = \dfrac{4}{(s+1)(s^2+2s+5)}$.
   (b) $G(s) = \dfrac{2s+3}{s^2+4}e^{-3s}$.

10. [**14 Points**] Let $A = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix}$.

    (a) Compute e^{At}.

    (b) Find the general solution of the system $\mathbf{y}' = A\mathbf{y}$.

    (c) Solve the initial value problem $\mathbf{y}' = A\mathbf{y}$, $\mathbf{y}(0) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

11. [**12 Points**]Let $f(t)$ be the periodic function of period 4 that is defined on the interval $(-2, 2]$ by

$$f(t) = \begin{cases} -1 & \text{if } -2 < t \le 0, \\ 2 & \text{if } 0 < t \le 2. \end{cases}$$

    (a) Sketch the graph of $f(t)$ on the interval $[-6, 6]$.

    (b) Compute the Fourier series of $f(t)$.

    (c) Let $g(t)$ denote the sum of the Fourier series found in part (b). Compute $g(0)$ and $g(7)$.

12. [**14 Points**] A 200-liter tank initially contains 100 L of brine with a concentration of 3 g/L of salt (i.e., 3 grams of salt per liter of water). Starting at time $t = 0$, brine with a salt concentration of 5 g/L runs into the tank at the rate of 8 L/min. The well-mixed solution is drawn off at the rate of 6 L/min. Let $y(t)$ denote the number of grams of salt in the tank at time t.

    (a) What is $y(0)$?

    (b) What the volume $V(t)$ of water in the tank at time t? At what time will the tank overflow?

    (c) What is the differential equation that $y(t)$ satisfies until the tank overflows?

    (d) Solve this differential equation to determine the amount of salt in the tank at time t, up until the tank overflows.

    (e) Find the amount of salt in the tank at the moment that the tank overflows.

Laplace Transform Table

	$f(t)$	$\rightarrow$	$F(s) = \mathcal{L}\left\{f(t)\right\}(s)$
1.	1	$\rightarrow$	$\dfrac{1}{s}$
2.	t^n	$\rightarrow$	$\dfrac{n!}{s^{n+1}}$
3.	e^{at}	$\rightarrow$	$\dfrac{1}{s-a}$
4.	$t^n e^{at}$	$\rightarrow$	$\dfrac{n!}{(s-a)^{n+1}}$
5.	$\cos bt$	$\rightarrow$	$\dfrac{s}{s^2+b^2}$
6.	$\sin bt$	$\rightarrow$	$\dfrac{b}{s^2+b^2}$
7.	$e^{at}\cos bt$	$\rightarrow$	$\dfrac{s-a}{(s-a)^2+b^2}$
8.	$e^{at}\sin bt$	$\rightarrow$	$\dfrac{b}{(s-a)^2+b^2}$
9.	$h(t-c)$	$\rightarrow$	$\dfrac{e^{-sc}}{s}$
10.	$\delta_c(t)$	$\rightarrow$	e^{-sc}

Laplace Transform Principles

Linearity	$\mathcal{L}\left\{af(t)+bg(t)\right\}$	$=$	$a\mathcal{L}\left\{f\right\}+b\mathcal{L}\left\{g\right\}$
Input Derivative Principles	$\mathcal{L}\left\{f'(t)\right\}(s)$	$=$	$s\mathcal{L}\left\{f(t)\right\}-f(0)$
	$\mathcal{L}\left\{f''(t)\right\}(s)$	$=$	$s^2\mathcal{L}\left\{f(t)\right\}-sf(0)-f'(0)$
First Translation Principle	$\mathcal{L}\left\{e^{at}f(t)\right\}$	$=$	$F(s-a)$
Transform Derivative Principle	$\mathcal{L}\left\{-tf(t)\right\}(s)$	$=$	$\dfrac{d}{ds}F(s)$
Second Translation Principle	$\mathcal{L}\left\{h(t-c)f(t-c)\right\}$	$=$	$e^{-sc}F(s)$, or
	$\mathcal{L}\left\{g(t)h(t-c)\right\}$	$=$	$e^{-sc}\mathcal{L}\left\{g(t+c)\right\}$.
The Convolution Principle	$\mathcal{L}\left\{(f*g)(t)\right\}(s)$	$=$	$F(s)G(s)$.

Partial Fraction Expansion Theorems

The following two theorems are the main partial fractions expansion theorems, as presented in the text.

Theorem 1 (Linear Case). *Suppose a proper rational function can be written in the form*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)}$$

and $q(\lambda) \neq 0$. *Then there is a unique number* A_1 *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)} = \frac{A_1}{(s-\lambda)^n} + \frac{p_1(s)}{(s-\lambda)^{n-1} q(s)}. \tag{1}$$

The number A_1 *and the polynomial* $p_1(s)$ *are given by*

$$A_1 = \frac{p_0(\lambda)}{q(\lambda)} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - A_1 q(s)}{s-\lambda}. \tag{2}$$

Theorem 2 (Irreducible Quadratic Case). *Suppose a real proper rational function can be written in the form*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)},$$

where s^2+cs+d *is an irreducible quadratic that is factored completely out of* $q(s)$. *Then there is a unique linear term* $B_1 s + C_1$ *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)} = \frac{B_1 s + C_1}{(s^2+cs+d)^n} + \frac{p_1(s)}{(s^s+cs+d)^{n-1} q(s)}. \tag{3}$$

If $a+ib$ *is a complex root of* s^2+cs+d *then* $B_1 s + C_1$ *and the polynomial* $p_1(s)$ *are given by*

$$B_1(a+ib) + C_1 = \frac{p_0(a+ib)}{q(a+ib)} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - (B_1 s + C_1) q(s)}{s^2+cs+d}. \tag{4}$$