Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* A table of Laplace transforms and the statement of the partial fraction decomposition theorems are attached to the exam.

In Exercises 1 – 6, solve the given differential equation. If initial values are given, solve the initial value problem. Otherwise, give the general solution. Some problems may be solvable by more than one technique. You are free to choose whatever technique that you deem to be most appropriate.

1. [**12 Points**] $y' + 2y = e^{2t} + e^{-2t}, \quad y(0) = -1.$

   ▶ **Solution.** This equation is linear. To find an integrating factor note that $p(t) = 2$ so that an integrating factor is $\mu(t) = e^{\int p(t)\,dt} = e^{2t}$. Multiplying the equation by e^{2t} gives
   $$e^{2t}y' + 2e^{2t}y = e^{4t} + 1.$$
   Thus,
   $$\left(e^{2t}y\right)' = e^{4t} + 1,$$
   and integrating gives
   $$e^{2t}y = \frac{1}{4}e^{4t} + t + C.$$
   Solving for y gives
   $$y = \frac{1}{4}e^{2t} + te^{-2t} + Ce^{-2t}.$$
   Now apply the initial condition $y(0) = -1$ to get $-1 = y(0) = \frac{1}{4} + C$. Therefore, $C = -5/4$ and
   $$\boxed{y(t) = \frac{1}{4}e^{2t} + te^{-2t} - \frac{5}{4}e^{-2t}.}$$
   ◀

2. [**9 Points**] $y'' + 4y' + 13y = 0.$

   ▶ **Solution.** The characteristic polynomial is $q(s) = s^2 + 4s + 13 = (s+2)^2 + 9$, which has roots $-2 \pm 3i$. Thus, the solutions are given by
   $$\boxed{y(t) = c_1e^{-2t}\cos 3t + c_2e^{-2t}\sin 3t.}$$
   ◀

3. [**9 Points**] $4y'' + 20y' + 25y = 0$. The characteristic polynomial is $q(s) = 4s^2 + 20s + 25 = (2s+5)^2$, which has a single root $-5/2$ of multiplicity 2. Thus, the solutions are given by
   $$\boxed{y(t) = c_1e^{-5t/2} + c_2te^{-5t/2}.}$$

4. [**12 Points**] $3y'' + 11y' + 10y = 0, \quad y(0) = 0,\ y'(0) = 2.$

   ▶ **Solution.** The characteristic polynomial is $q(s) = 3s^2 + 11s + 10 = (3s+5)(s+2)$, which has roots $-5/3$ and -2. Thus, the general solution of the differential equation is $y(t) = c_1 e^{-5t/3} + c_2 e^{-2t}$. The initial conditions give the equations for c_1, c_2:

$$
\begin{aligned}
0 &= y(0) = c_1 + c_2 \\
2 &= y'(0) = -\frac{5}{3}c_1 - 2c_2.
\end{aligned}
$$

   Solve these equations to get $c_1 = 6$, $c_2 = -6$. Thus, the solution of the initial value problem is

$$\boxed{y(t) = 6e^{-5t/2} - 6e^{-2t}.}$$

◀

5. [**12 Points**] $2t^2y'' + 5ty' - 2y = 0, \qquad y(1) = -1,\ y'(1) = 3.$

   ▶ **Solution.** This is a Cauchy-Euler equation with indicial polynomial $Q(s) = 2s(s-1)+5s-2 = 2s^2+3s-2 = (2s-1)(s+2)$ that has roots $1/2$ and -2. Thus, the general solution of the differential equation is $y(t) = c_1 t^{1/2} + c_2 t^{-2}$. The initial equations give the equations for c_1, c_2:

$$
\begin{aligned}
-1 &= y(1) = c_1 + c_2 \\
3 &= y'(1) = \frac{1}{2}c_1 - 2c_2.
\end{aligned}
$$

   Solve these equations to get $c_1 = \frac{2}{5}$ and $c_2 = -\frac{7}{5}$. Thus, the solution of the initial value problem is

$$\boxed{y(t) = \frac{2}{5}t^{1/2} - \frac{7}{5}t^{-2}.}$$

◀

6. [**12 Points**] $y'' + 2y' + y = 2\sin 3t.$

   ▶ **Solution.** Use the method of undetermined coefficients. The characteristic polynomial of the associated homogeneous equation is $q(s) = s^2 + 2s + 1 = (s+1)^2$. The roots are -1 with multiplicity 2 so that the standard basis is $\mathcal{B}_q = \{e^{-t},\ te^{-t}\}$. The Laplace transform of the right hand side is $\mathcal{L}\{2\sin 3t\} = \dfrac{6}{s^2+9}$, which has denominator $v(s) = s^+9$. Thus, $q(s)v(s) = (s+1)^2(s^2+9)$ and

$$\mathcal{B}_{qv} \setminus \mathcal{B}_q = \left\{e^{-t},\ te^{-t},\ \sin 3t,\ \cos 3t\right\} \setminus \left\{e^{-t},\ te^{-t}\right\} = \{\sin 3t,\ \cos 3t\}.$$

Thus, a particular solution of the nonhomogeneous equation will have the form $y_p(t) = A\sin 3t + B\cos 3t$, and the unknown constants A and B can be determined by substituting this back in the differential equation. Compute the derivatives of $y_p(t)$:

$$\begin{aligned} y_p &= A\sin 3t + B\cos 3t \\ y_p' &= 3A\cos 3t - 3B\sin 3t \\ y_p'' &= -9A\sin 3t - 9B\cos 3t \end{aligned}$$

Substituting in the differential equation gives

$$2\sin 3t = y_p'' + 2y_p' + y_p = (-8A - 6B)\sin 3t + (-8B + 6A)\cos 3t.$$

Equating the coefficients of $\sin 3t$ and $\cos 3t$ on the left and right gives the equations

$$\begin{aligned} -8A - 6B &= 2 \\ 6A - 8B &= 0. \end{aligned}$$

Solving for A and B gives $A = -\dfrac{4}{25}$ and $B = -\dfrac{3}{25}$. Thus, $y_p(t) = -\frac{1}{25}(4\sin 3t + 3\cos 3t)$ and the general solution is

$$\boxed{y(t) = y_h(t) + y_p(t) = c_1e^{-t} + c_2te^{-t} - \frac{4}{25}\sin 3t - \frac{3}{25}\cos 3t.}$$

◀

7. [**12 Points**] Find the general solution for $t > 0$ of the differential equation

$$t^2y'' + 2ty' - 2y = \sqrt{t},$$

given the fact that two solutions of the associated homogeneous equation are $y_1(t) = t$ and $y_2(t) = t^{-2}$.

▶ **Solution.** Use variation of parameters. First divide by t^2 to put the equation in standard form

$$y'' + \frac{2}{t}y' - \frac{2}{t^2}y = 3t = t^{-3/2}.$$

A particular solution is given by

$$y_p = u_1t + u_2t^{-2}$$

where u_1' and u_2' satisfy the equations:

$$\begin{aligned} u_1't + u_2't^{-2} &= 0 \\ u_t' - 2u_2't^{-3} &= t^{-3/2}. \end{aligned}$$

Thus, subtracting t times the second equation from the first gives

$$3u_2't^{-2} = -t^{-1/2}$$

so that $u_2' = -\frac{1}{3}t^{3/2}$ and then

$$u_1' = -t^{-3}u_2' = \frac{1}{3}t^{-3/2}.$$

Integrating gives $u_1 = -\frac{2}{3}t^{-1/2}$ and $u_2 = -\frac{2}{15}t^{5/2}$ and

$$\begin{aligned} y_p &= u_1 t + u_2 t^{-2} \\ &= -\frac{2}{3}t^{-1/2}t - \frac{2}{15}t^{5/2}t^{-2} \\ &= -\left(\frac{2}{3} + \frac{2}{15}\right)t^{1/2} \\ &= -\frac{4}{5}t^{1/2}. \end{aligned}$$

Then the general solution is $y = y_h + y_p$ so

$$\boxed{y = c_1 t + c_2 t^{-2} - \frac{4}{5}t^{1/2}.}$$

◀

8. [**16 Points**] Let $f(t)$ be the following function:

$$f(t) = \begin{cases} 1 & \text{if } 0 \le t < 2, \\ t-1 & \text{if } t \ge 2. \end{cases}$$

(a) Compute the Laplace transform $F(s)$ of $f(t)$.

▶ **Solution.** Write $f(t)$ in terms of the Heaviside step function:

$$\begin{aligned} f(t) &= \chi_{[0,2)}9(t) + (t-1)\chi_{[2,\infty)}(t) \\ &= h(t) - h(t-1) + (t-1)h(t-2) \\ &= 1 + (t-2)h(t-2). \end{aligned}$$

Then the second translation theorem gives

$$\boxed{F(s) = \frac{1}{s} + \frac{e^{-2s}}{s^2}.}$$

◀

(b) Find the Laplace transform $Y(s)$ of the solution of the initial value problem

$$y'' + 2y' + 5y = f(t), \qquad y(0) = 2,\ y'(0) = -1,$$

DO NOT solve for $y(t)$. Just find $Y(s)$. You may express your answer in terms of $F(s)$.

▶ **Solution.** Apply the Laplace transform to both sides of the equation to get

$$s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + 5Y(s) = F(s).$$

Thus,

$$(s^2 + 2s + 5)Y(s) - 2s + 1 - 4 = F(s),$$

so

$$\boxed{Y(s) = \frac{2s + 3 + F(s)}{s^2 + 2s + 5}.}$$

◀

9. [**16 Points**] Compute the inverse Laplace transform of the following functions:

(a) $F(s) = \dfrac{4}{(s+1)(s^2 + 2s + 5)}$.

▶ **Solution.** First expand $F(s)$ in a partial fraction expansion:

$$F(s) = \frac{A}{s+1} + \frac{p_1(s)}{s^2 + 2s + 5}$$

where

$$A = \left.\frac{4}{s^2 + 2s + 5}\right|_{s=-1} = \frac{4}{4} = 1$$

and

$$p_1(s) = \frac{4 - (s^2 + 2s + 5)}{s+1} = \frac{-s^2 - 2s - 1}{s+1} = -\frac{(s+1)^2}{s+1} = -(s+1).$$

Thus

$$F(s) = \frac{1}{s+1} - \frac{s+1}{(s+1)^2 + 1}.$$

Then

$$\boxed{f(t) = \mathcal{L}^{-1}\left\{F(s)\right\} = e^{-t} - e^{-t}\cos 2t.}$$

◀

(b) $G(s) = \dfrac{2s+3}{s^2+4}e^{-3s}$.

▶ **Solution.** Since $\dfrac{2s+3}{s^2+4} = 2\dfrac{2}{s^2+4} + 3\dfrac{1}{s^2+4}$, we have

$$\mathcal{L}^{-1}\left\{\frac{2s+3}{s^2+4}\right\} = 2\cos 2t + \frac{3}{2}\sin 2t,$$

and the second translation theorem gives

$$\boxed{\mathcal{L}^{-1}\left\{\frac{2s+3}{s^2+4}e^{-3s}\right\} = \big(2\cos 2(t-3) + \frac{3}{2}\sin 2(t-3)\big)h(t-3).}$$

◀

10. [**14 Points**] Let $A = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix}$.

(a) Compute e^{At}.

▶ **Solution.** Use Fulmer's method. $sI - A = \begin{bmatrix} s & -1 \\ 9 & s-6 \end{bmatrix}$ so $q(s) = \det(sI - A) = s(s-6)+9 = s^2 - 6s + 9 = (s-3)^2$. Then $\mathcal{B}_q = \{e^{3t},\ te^{3t}\}$ so $e^{At} = M_1 e^{3t} + M_2 te^{3t}$. Differentiating gives $Ae^{At} = 3M_1e^{3t} + M_2(e^{3t} + 3te^{3t})$, and evaluating both of these at $t = 0$ gives the equations

$$\begin{aligned} I &= M_1 \\ A &= 3M_1 + M_2. \end{aligned}$$

Thus, $M_1 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $M_2 = A - 3I = \begin{bmatrix} -3 & 1 \\ -9 & 3 \end{bmatrix}$. Hence

$$\begin{aligned} e^{At} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} e^{3t} + \begin{bmatrix} -3 & 1 \\ -9 & 3 \end{bmatrix} te^{3t} \\ &= \begin{bmatrix} e^{3t} - 3te^{3t} & te^{3t} \\ -9te^{3}t & e^{3t} + 3te^{3t} \end{bmatrix}. \end{aligned}$$

◀

(b) Find the general solution of the system $\mathbf{y}' = A\mathbf{y}$.

▶ **Solution.** The general solution is $\mathbf{y}(t) = e^{At}\mathbf{y}(0)$ where $\mathbf{y}(0) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. Thus,

$$\begin{aligned} \mathbf{y}(t) &= e^{At} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} e^{3t} - 3te^{3t} & te^{3t} \\ -9te^{3}t & e^{3t} + 3te^{3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} c_1(e^{3t} - 3te^{3t} + c_2te^{3t} \\ c_1(-9te^{3t}) + c_2(e^{3t} + 3te^{3t}) \end{bmatrix} = \begin{bmatrix} c_1e^{3t} + (c_2 - 3c_1)te^{3t} \\ c_2e^{3t} + (3c_2 - 9c_1)te^{3t} \end{bmatrix}. \end{aligned}$$

◀

(c) Solve the initial value problem $\mathbf{y}' = A\mathbf{y}$, $\mathbf{y}(0) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

▶ **Solution.** Setting $c_1 = 3$, $c_2 = -2$ in part (b) gives

$$\mathbf{y}(t) = \begin{bmatrix} 3e^{3t} - 11te^{3t} \\ -2e^{3t} - 33e^{3t} \end{bmatrix}.$$

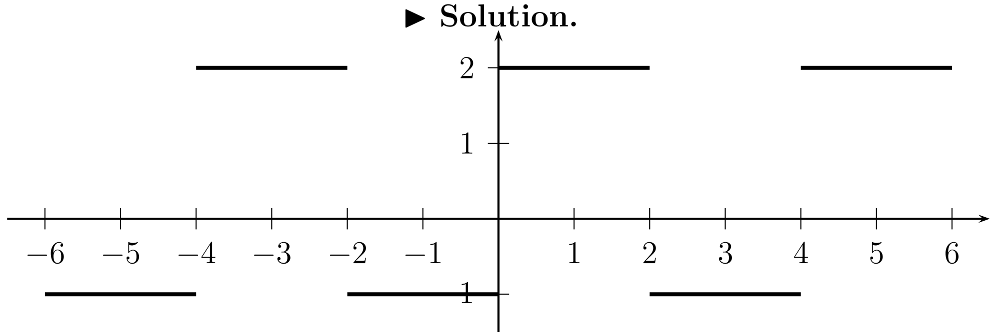
◀

11. [**12 Points**]Let $f(t)$ be the periodic function of period 4 that is defined on the interval $(-2, 2]$ by

$$f(t) = \begin{cases} -1 & \text{if } -2 < t \le 0, \\ 2 & \text{if } 0 < t \le 2. \end{cases}$$

(a) Sketch the graph of $f(t)$ on the interval $[-6, 6]$.

▶ **Solution.**



◀

(b) Compute the Fourier series of $f(t)$.

▶ **Solution.** Since the period is 4, the half-period is 2 so the Fourier series has the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi}{2}x + b_n \sin\frac{n\pi}{2}x)$$

where the coefficients are computed by:

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\int_{-2}^{0} -\,dx + \frac{1}{2}\int_{0}^{2} 2\,dx = \frac{1}{2}(-2+4) = 1.$$

For $n \ge 1$

$$\begin{aligned} a_n &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi}{2}x\,dx \\ &= \frac{1}{2}\int_{-2}^{0} -\cos\frac{n\pi}{2}x\,dx + \frac{1}{2}\int_{0}^{2} 2\cos\frac{n\pi}{2}x\,dx \\ &= \frac{1}{2}\int_{0}^{2} -\cos\frac{n\pi}{2}x\,dx + \frac{1}{2}\int_{0}^{2} 2\cos\frac{n\pi}{2}x\,dx \\ &= \frac{1}{2}\int_{0}^{2} \cos\frac{n\pi}{2}x\,dx \\ &= \frac{1}{n\pi}\sin\frac{n\pi}{2}x\Big|_0^2 \\ &= 0. \end{aligned}$$

and

$$\begin{aligned}
b_n &= \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi}{2}x\,dx \\
&= \frac{1}{2}\int_{-2}^{0} -\sin\frac{n\pi}{2}x\,dx + \frac{1}{2}\int_{0}^{2} 2\sin\frac{n\pi}{2}x\,dx \\
&= \frac{1}{2}\int_{0}^{2} \sin\frac{n\pi}{2}x\,dx + \frac{1}{2}\int_{0}^{2} 2\sin\frac{n\pi}{2}x\,dx \\
&= \frac{3}{2}\int_{0}^{2} \sin\frac{n\pi}{2}x\,dx \\
&= -\frac{3}{2}\cdot\frac{2}{n\pi}\cos\frac{n\pi}{2}x\Big|_0^2 \\
&= -\frac{3}{n\pi}(\cos n\pi - 1) \\
&= -\frac{3}{n\pi}((-1)^n - 1).
\end{aligned}$$

Thus, the Fourier series of $f(x)$ is

$$\begin{aligned}
f(x) &\sim \frac{1}{2} - \frac{3}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n}\sin\frac{n\pi}{2}x \\
&\sim \frac{1}{2} + \frac{6}{\pi}\sum_{n \text{ odd}}\frac{1}{n}\sin\frac{n\pi}{2}x.
\end{aligned}$$

◀

(c) Let $g(t)$ denote the sum of the Fourier series found in part (b). Compute $g(0)$ and $g(7)$.

▶ **Solution.** $g(0) = \frac{2+(-1)}{2} = \frac{1}{2}$ and $g(7) = g(7-8) = g(-1) = -1$. ◀

12. [**14 Points**] A 200-liter tank initially contains 100 L of brine with a concentration of 3 g/L of salt (i.e., 3 grams of salt per liter of water). Starting at time $t = 0$, brine with a salt concentration of 5 g/L runs into the tank at the rate of 8 L/min. The well-mixed solution is drawn off at the rate of 6 L/min. Let $y(t)$ denote the number of grams of salt in the tank at time t.

(a) What is $y(0)$?

▶ **Solution.** $y(0) = 100 \text{ L} \cdot 3\text{g/L} = 300\text{g}$. ◀

(b) What the volume $V(t)$ of water in the tank at time t? At what time will the tank overflow?

▶ **Solution.** $V(t) = 100 + (8-6)t = 100 + 2t$. The tank will overflow when $V(t) = 200$, i.e. when $t = 50$. ◀

(c) What is the differential equation that $y(t)$ satisfies until the tank overflows?

▶ **Solution.** The balance equation is

$$y'(t) = \text{rate in} - \text{rate out}.$$

The rate in is 8 L/min times the concentration of 5 g/L. Thus the rate in is 40 g/min.The rate out is

$$\left(\frac{y(t)}{V(t)}\right) \times 6.$$

From part (b) $V(t) = 100 + 2t$. Hence $y(t)$ satisfies the equation

$$y' = 40 - \frac{6}{100 + 2t} y,$$

which can be written in standard form as

$$y' + \frac{6}{100 + 2t} y = 40.$$

◀

(d) Solve this differential equation to determine the amount of salt in the tank at time t, up until the tank overflows.

▶ **Solution.** This is a linear differential equation with $p(t) = 6/(100+2t)$ so that $P(t) = \int p(t)\, dt = 6 \cdot \frac{1}{2} \ln(100 + 2t) = \ln(100 + 2t)^3$. Therefore the integrating factor is $\mu(t) = e^{\ln(100+2t)^3} = (100 + 2t)^3$, so multiplication of the differential equation by $\mu(t)$ gives an equation

$$\frac{d}{dt}\left((100 + 2t)^3 y\right) = 40(100 + 2t)^3.$$

Integration of this equation gives

$$(100 + 2t)^3 y = 5(100 + 2t)^4 + C,$$

where C is an integration constant. Thus

$$y(t) = 5(100 + 2t) + C(100 + 2t)^{-3}$$

and the initial condition $y(0) = 300$ gives

$$300 = y(0) = 500 + C100^{-3}$$

so that $C = -200 \cdot 100^3 = -2(100)^4$, and

$$y(t) = 5(100 + 2t) - 200 \left(\frac{100}{100 + 2t}\right)^3.$$

◀

(e) Find the amount of salt in the tank at the moment that the tank overflows.

► **Solution.** The tank will overflow at $t = 50$ so putting $t = 50$ in the above solution will give

$$\begin{aligned} y(50) &= 5(1000) - 200\left(\frac{100}{200}\right)^3 \\ &= 1000 - 25 = 975\text{g}. \end{aligned}$$

◄

Laplace Transform Table

	$f(t)$	$\longleftrightarrow$	$F(s) = \mathcal{L}\{f(t)\}(s)$
1.	1	$\longleftrightarrow$	$\dfrac{1}{s}$
2.	t^n	$\longleftrightarrow$	$\dfrac{n!}{s^{n+1}}$
3.	e^{at}	$\longleftrightarrow$	$\dfrac{1}{s-a}$
4.	$t^n e^{at}$	$\longleftrightarrow$	$\dfrac{n!}{(s-a)^{n+1}}$
5.	$\cos bt$	$\longleftrightarrow$	$\dfrac{s}{s^2+b^2}$
6.	$\sin bt$	$\longleftrightarrow$	$\dfrac{b}{s^2+b^2}$
7.	$e^{at}\cos bt$	$\longleftrightarrow$	$\dfrac{s-a}{(s-a)^2+b^2}$
8.	$e^{at}\sin bt$	$\longleftrightarrow$	$\dfrac{b}{(s-a)^2+b^2}$
9.	$h(t-c)$	$\longleftrightarrow$	$\dfrac{e^{-sc}}{s}$
10.	$\delta_c(t)$	$\longleftrightarrow$	e^{-sc}

Laplace Transform Principles

Linearity	$\mathcal{L}\{af(t)+bg(t)\}$	$=$	$a\mathcal{L}\{f\}+b\mathcal{L}\{g\}$
Input Derivative Principles	$\mathcal{L}\{f'(t)\}(s)$	$=$	$s\mathcal{L}\{f(t)\}-f(0)$
	$\mathcal{L}\{f''(t)\}(s)$	$=$	$s^2\mathcal{L}\{f(t)\}-sf(0)-f'(0)$
First Translation Principle	$\mathcal{L}\{e^{at}f(t)\}$	$=$	$F(s-a)$
Transform Derivative Principle	$\mathcal{L}\{-tf(t)\}(s)$	$=$	$\dfrac{d}{ds}F(s)$
Second Translation Principle	$\mathcal{L}\{h(t-c)f(t-c)\}$	$=$	$e^{-sc}F(s)$, or
	$\mathcal{L}\{g(t)h(t-c)\}$	$=$	$e^{-sc}\mathcal{L}\{g(t+c)\}$.
The Convolution Principle	$\mathcal{L}\{(f*g)(t)\}(s)$	$=$	$F(s)G(s)$.

Partial Fraction Expansion Theorems

The following two theorems are the main partial fractions expansion theorems, as presented in the text.

Theorem 1 (Linear Case). *Suppose a proper rational function can be written in the form*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)}$$

and $q(\lambda) \neq 0$. *Then there is a unique number* A_1 *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)} = \frac{A_1}{(s-\lambda)^n} + \frac{p_1(s)}{(s-\lambda)^{n-1} q(s)}. \tag{1}$$

The number A_1 *and the polynomial* $p_1(s)$ *are given by*

$$A_1 = \frac{p_0(\lambda)}{q(\lambda)} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - A_1 q(s)}{s-\lambda}. \tag{2}$$

Theorem 2 (Irreducible Quadratic Case). *Suppose a real proper rational function can be written in the form*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)},$$

where s^2+cs+d *is an irreducible quadratic that is factored completely out of* $q(s)$. *Then there is a unique linear term* $B_1 s + C_1$ *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)} = \frac{B_1 s + C_1}{(s^2+cs+d)^n} + \frac{p_1(s)}{(s^s+cs+d)^{n-1} q(s)}. \tag{3}$$

If $a+ib$ *is a complex root of* s^2+cs+d *then* $B_1 s + C_1$ *and the polynomial* $p_1(s)$ *are given by*

$$B_1(a+ib) + C_1 = \frac{p_0(a+ib)}{q(a+ib)} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - (B_1 s + C_1) q(s)}{s^2+cs+d}. \tag{4}$$