Instructions. Answer each of the questions on your own paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* Put your name on each page of your paper.

1. [**8 Points**] Determine if each of the following first order differential equations is separable (**Yes** or **No**), and/or linear (**Yes** or **No**). Do **not** solve the equations.

Equation	Separable	Linear
$y' + y = t^2$	**No**	**Yes**
$y' + y^2 = t^2$	**No**	**No**
$y' + y^2 t = 0$	**Yes**	**No**
$y' + t^2 y = t^2$	**Yes**	**Yes**

2. [**12 Points**] Let $A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$. Compute each of the following matrices, if it exists.

(a) AB (b) BA (c) A^2 (d) B^2

▶ **Solution.** (a) AB does not exist.

(b) $BA = \begin{bmatrix} -1 & 1 & -2 \\ 7 & 2 & -1 \end{bmatrix}$.

(c) A^2 does not exist.

(d) $B^2 = \begin{bmatrix} -1 & -2 \\ 4 & -1 \end{bmatrix}$.

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3. [**16 Points**] Find all solutions to the linear system

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & + & x_3 & + & 3x_4 & = & 1 \\ x_1 & + & 2x_2 & + & 2x_3 & + & 5x_4 & = & 0 \\ -x_1 & - & 2x_2 & - & 3x_3 & - & 7x_4 & = & 1 \end{array}$$

▶ **Solution.** The augmented matrix of the system is

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 1 \\ 1 & 2 & 2 & 5 & 0 \\ -1 & -2 & -3 & -7 & 1 \end{bmatrix}.$$

Row reduce this matrix:

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 1 \\ 1 & 2 & 2 & 5 & 0 \\ -1 & -2 & -3 & -7 & 1 \end{bmatrix} \xrightarrow[t_{13}(1)]{t_{12}(-1)} \begin{bmatrix} 1 & 2 & 1 & 3 & 1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & -2 & -4 & 2 \end{bmatrix}$$

$$\xrightarrow{t_{23}(2)} \begin{bmatrix} 1 & 2 & 1 & 3 & 1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{t_{21}(-1)} \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This last matrix is the augmented matrix of the system

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & & & + & x_4 & = & 2 \\ & & & & x_3 & + & 2x_4 & = & -1 \\ & & & & & - & 0 & = & 0 \end{array}$$

This system has leading variables x_1 and x_3 and free variables x_2 and x_4. Solving for x_1 and x_3 in terms of x_2 and x_4, and letting $x_2 = s$, $x_4 = t$, where s and t are arbitrary, gives the solutions

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2-2s-t \\ s \\ -1-2t \\ t \end{bmatrix}.$$

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4. [**16 Points**] Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$.

(a) Compute $\det A$.

▶ **Solution.** Using cofactor expansion along the first row:

$$\det A = 1 \cdot \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} - 0 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} = 4 - 2 = 2.$$

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(b) Compute the inverse of A.

▶ **Solution.**

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 2 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}$$

Thus,

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1/2 & -1/2 \\ -1 & 0 & 1 \end{bmatrix}.$$

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5. [**16 Points**] Solve the initial value problem: $y' + 5y = 30e^{-5t} + 30e^{5t}, \quad y(0) = -5.$

▶ **Solution.** This equation is linear with coefficient function $p(t) = 5$ so that an integrating factor is given by $\mu(t) = e^{\int 5\,dt} = e^{5t}$. Multiplication of the differential equation by the integrating factor gives

$$e^{5t}y' + 5e^{5t}y = 30 + 30e^{10t},$$

and the left hand side is recognized (by the choice of $\mu(t)$) as a perfect derivative:

$$\frac{d}{dt}(e^{5t}y) = 30 + 30e^{10t}.$$

Integration then gives

$$e^{5t}y = 30t + 3e^{10t} + C,$$

where C is an integration constant. Multiplying through by e^{-5t} gives

$$y = 30te^{-5t} + 3e^{5t} + Ce^{-5t}.$$

The initial condition gives $-5 = y(0) = 3 + C$ so $C = -8$ and thus

$$\boxed{y(t) = 30te^{-5t} + 3e^{5t} - 8e^{-5t}.}$$

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6. [**16 Points**] Solve the initial value problem: $y' = y^2(t+2), \quad y(0) = 1.$

▶ **Solution.** This equation is separable. There is one equilibrium solution: $y = 0$. This does not satisfy the initial condition so it is not the required solution. If $y \neq 0$, separate variables to get $y^{-2}y' = t + 2$, which in differential form is

$$y^{-2}\,dy = (t+2), dt$$

and integration then gives

$$-y^{-1} = \int (t+2)\,dt = \frac{1}{2}(t+2)^2 + C,$$

so that

$$y = \frac{-1}{\frac{1}{2}(t+2)^2 + C}.$$

The initial condition $y(0) = 1$ gives $1 = -1/(2 + C)$ so $C = -3$, and

$$\boxed{y = \frac{-1}{\frac{1}{2}(t+2)^2 - 3} = \frac{-2}{(t+2)^2 - 6}.}$$

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7. [**16 Points**] Solve the initial value problem: $y' + \dfrac{2}{t}y = t^4, \quad y(1) = \dfrac{22}{7}.$

▶ **Solution.** This equation is linear and already in standard form. Thus, the coefficient function is $p(t) = \frac{2}{t}$, so that an integrating factor is $\mu(t) = e^{\int \frac{2}{t}\,dt} = e^{2\ln|t|} = e^{\ln(|t|^2)} = |t|^2 = t^2$. Multiplication of the differential equation by the integrating factor gives

$$t^2y' + 2ty = t^6,$$

and the left hand side is recognized (by the choice of $\mu(t)$) as a perfect derivative:

$$\frac{d}{dt}(t^2y) = t^6.$$

Integration then gives

$$t^2y = \frac{t^7}{7} + C,$$

where C is an integration constant. Multiplying through by t^{-2} gives

$$y = \frac{t^5}{7} + Ct^{-2}.$$

The initial condition $y(1) = \dfrac{22}{7}$ implies $\dfrac{22}{7} = y(1) = \dfrac{1}{7} + C$ so that $C = 3$. Hence

$$\boxed{y = \frac{t^5}{7} + 3t^{-1}.}$$

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