Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* A table of Laplace transforms, a table of convolutions, and the statement of the main partial fraction decomposition theorem have been appended to the exam.

1. [**24 Points**] Compute the Laplace transform of each of the following functions. You may use the attached tables, but be sure to identify which formulas you are using by citing the number(s) or name of the formula in the table.

   (a) $f_1(t) = e^{-t/2}(\cos 2t - \sin \sqrt{5}t)$

   ▶ **Solution.** $f_1(t) = e^{-t/2}\cos 2t - e^{-t/2}\sin\sqrt{5}t)$ so

$$F_1(s) = \left[\frac{s}{s^2+4} - \frac{\sqrt{5}}{s^2+5}\right]\Bigg|_{s\mapsto s-(-\frac{1}{2})} = \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+4} - \frac{\sqrt{5}}{(s+\frac{1}{2})^2+5}.$$

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   (b) $f_2(t) = 5 + 3t\cos 6t$

   ▶ **Solution.** Since $\mathcal{L}\{3\cos 6t\} = \dfrac{3s}{s^2+6^2}$, the Transform Derivative Principle gives

$$F_2(s) = \frac{5}{s} - \frac{d}{ds}\left(\frac{3s}{s^2+6^2}\right) = \frac{5}{s} - \frac{3(s^2+6^2) - 3s\cdot 2s}{(s^2+6^2)^2} = \frac{5}{s} + \frac{3s^2-108}{(s^2+6^2)^2}.$$

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   (c) $f_3(t) = (\cos 2t) * g(t)$ where $g(t)$ is the function with Laplace transform

$$G(s) = \frac{3s^2}{(s^2+4)^2.}$$

   Recall that $f * g$ is the *convolution* product of f and g.

   ▶ **Solution.** From the convolution principle,

$$F_3(s) = (\mathcal{L}\{\cos 2t\})\,\mathcal{L}\{g(t)\} = \left(\frac{s}{s^2+4}\right)G(s) = \frac{s}{(s^2+4)}\cdot\frac{3s^2}{(s^2+4)^2} = \frac{3s^3}{(s^2+4)^3}.$$

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2. **[10 Points]** Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial value problem.

$$2y'' + 7y' + 3y = 3t^2, \qquad y(0) = 2,\ y'(0) = -3.$$

Note that you are asked to find $Y(s)$, but *not* $y(t)$.

▶ **Solution.** Apply the Laplace transform to the differential equation, using linearity and the input derivative principle to get

$$2(s^2Y(s) - sy(0) - y'(0)) + 7(sY(s) - y(0)) + 3Y(s) = \frac{6}{s^3}.$$

This gives

$$2s^2Y(s) - 4s + 6 + 7sY(s) - 14 + 3Y(s) = \frac{6}{s^3}$$

or

$$(2s^2 + 7s + 3)Y(s) - 4s - 8 = \frac{6}{s^3}.$$

Solve for $Y(s)$ to get

$$Y(s) = \frac{4s+8}{2s^2+7s+3} + \frac{6}{s^3(2s^2+7s+3)}.$$

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3. **[16 Points]** Compute the inverse Laplace transform of each of the following rational functions.

(a) $F(s) = \dfrac{2s^2+3}{s^3+2s^2-3s}$

▶ **Solution.** First factor the denominator to get

$$F(s) = \frac{2s^2+3}{s(s^2+2s-3)} = \frac{2s^2+3}{s(s+3)(s-1)},$$

and then expand $F(s)$ into partial fractions. Since the denominator is a product of distinct linear terms,

$$F(s) = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s-1},$$

where

$$A = \left.\frac{2s^2+3}{(s+3)(s-1)}\right|_{s=0} = \frac{3}{-3} = -1,$$

$$B = \left.\frac{2s^2+3}{s(s-1)}\right|_{s=-3} = \frac{2\cdot 9+3}{(-3)(-4)} = \frac{21}{12} = \frac{7}{4},$$

$$C = \left.\frac{2s^2+3}{s(s+3)}\right|_{s=1} = \frac{5}{4}.$$

Thus,

$$F(s) = \frac{-1}{s} + \frac{7/4}{s+3} + \frac{5/4}{s-1},$$

so that

$$f(t) = \mathcal{L}^{-1}\left\{F(s)\right\} = -1 + \frac{7}{4}e^{-3t} + \frac{5}{4}e^{t}.$$

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(b) $G(s) = \dfrac{3s+7}{s^2+6s+25}$

▶ **Solution.**

$$\begin{aligned} G(s) = \frac{3s+7}{s^2+6s+25} &= \frac{3s+7}{(s+3)^2+16} \\ &= \frac{3((s+3)-3)+7}{(s+3)^2+16} = \frac{3(s+3)-2}{(s+4)^2+9} \\ &= \frac{3(s+3)}{(s+3)^2+16} - \frac{2}{(s+3)^2+16}. \end{aligned}$$

Thus

$$g(t) = \mathcal{L}^{-1}\left\{G(s)\right\} = 3e^{-3t}\cos 4t - \frac{1}{2}e^{-3t}\sin 4t.$$

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4. **[35 Points]** Find the general solution of each of the following constant coefficient homogeneous differential equations.

(a) $4y'' + 9y = 0$

▶ **Solution.** The characteristic polynomial is $q(s) = 4s^2+9$ which has roots $\pm\frac{3}{2}i$. Thus

$$y = c_1 \cos\frac{3}{2}t + c_2 \sin\frac{3}{2}t.$$

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(b) $y'' + 2y' - 15y = 0$

▶ **Solution.** $q(s) = s^2 + 2s - 15 = (s+5)(s-3)$. The roots of $q(s)$ are -5 and 3. Thus

$$y = c_1 e^{-5t} + c_2 e^{3t}.$$

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(c) $y'' + 6y' + 9y = 0$

► **Solution.** $q(s) = s^2 + 6s + 9 = (s+3)^2$. There is a single root -3 with multiplicity 2. Thus

$$y = c_1 e^{-3t} + c_2 t e^{-3t}.$$

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(d) $y'' + 6y' + 13y = 0$

► **Solution.** $q(s) = s^2 + 6s + 13 = (s+3)^2 + 4$. The roots of $q(s)$ are $-3 \pm 2i$. Thus

$$y = e^{-3t} \cos 2t + c_2 e^{-3t} \sin 2t.$$

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(e) $y''' + 2y'' + y' = 0$

► **Solution.** $q(s) = s^3 + 2s^2 + s = s(s^2 + 2s + 1) = s(s+1)^2$. Thus, $q(s)$ has a root 0 of multiplicity 1 and a root -1 of multiplicity 2, and hence

$$y = c_1 + c_2 e^{-t} + c_3 t e^{-t}.$$

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5. [**15 Points**] Find the general solution of the following differential equation:

$$y'' + 8y' + 16y = 4 \sin 2t.$$

You may use whatever method you prefer.

► **Solution.** Use the method of undetermined coefficients. The characteristic polynomial is $q(s) = s^2 + 8s16 = (s+4)^2$ which has a single root -4 of multiplicity 2. Thus $\mathcal{B}_q = \{e^{-4t}, te^{-4t}\}$ and $y_h = c_1 e^{-4t} + c_2 t e^{-4t}$. Since $\mathcal{L}\{4 \sin 2t\} = \dfrac{8}{s^2+4}$ the denominator is $v = s^2 + 4$ and $qv = (s+4)^2(s^2+4)$. Hence,

$$\mathcal{B}_{qv} \setminus \mathcal{B}_q = \left\{e^{-4t}, te^{-4t}, \cos 2t, \sin 2t\right\} \setminus \left\{e^{-4t}, te^{-4t}\right\} = \{\cos 2t, \sin 2t\}.$$

Therefore, the test function for y_p is $y_p = A \cos 2t + B \sin 2t$. Compute the derivatives:

$$\begin{aligned} y_p' &= -2A \sin 2t + 2B \cos 2t \\ y_p'' &= -4A \cos 2t - 4B \sin 2t. \end{aligned}$$

Substituting into the differential equation gives

$$\begin{aligned} 4 \sin 2t &= y_p'' + 8y_p' + 16y_p \\ &= (-4A \cos 2t - 4B \sin 2t) + 8(-2A \sin 2t + 2B \cos 2t) + 16(A \cos 2t + B \sin 2t) \\ &= (12A + 16B) \cos 2t + (-16A + 12B) \sin 2t. \end{aligned}$$

Comparing the coefficients of $\cos 2t$ and $\sin 2t$ on both sides of this equation shows that A and B satisfy the system of linear equations

$$\begin{aligned} 12A + 16B &= 0 \\ -16A + 12B &= 4. \end{aligned}$$

Dividing by 4 gives

$$\begin{aligned} 3A + 4B &= 0 \\ -4A + 3B &= 1. \end{aligned}$$

Solving this system by Cramer's rule gives

$$A = \frac{\begin{vmatrix} 0 & 4 \\ 1 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ -4 & 3 \end{vmatrix}} = \frac{-4}{25},$$

and

$$B = \frac{\begin{vmatrix} 3 & 0 \\ -4 & 1 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ -4 & 3 \end{vmatrix}} = \frac{3}{25}.$$

Thus,

$$y_p = -\frac{4}{25}\cos 2t + \frac{3}{25}\sin 2t,$$

and

$$y_g = y_h + y_p = c_1 e^{-4t} + c_2 t e^{-4t} - \frac{4}{25}\cos 2t + \frac{3}{25}\sin 2t.$$

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Laplace Transform Table

	$f(t)$	$\to$	$F(s) = \mathcal{L}\{f(t)\}(s)$
1.	1	$\to$	$\frac{1}{s}$
2.	t^n	$\to$	$\frac{n!}{s^{n+1}}$
3.	e^{at}	$\to$	$\frac{1}{s-a}$
4.	$t^n e^{at}$	$\to$	$\frac{n!}{(s-a)^{n+1}}$
5.	$\cos bt$	$\to$	$\frac{s}{s^2+b^2}$
6.	$\sin bt$	$\to$	$\frac{b}{s^2+b^2}$
7.	$e^{at}\cos bt$	$\to$	$\frac{s-a}{(s-a)^2+b^2}$
8.	$e^{at}\sin bt$	$\to$	$\frac{b}{(s-a)^2+b^2}$
9.	$\frac{1}{2b^2}(\sin bt - bt\cos bt)$	$\to$	$\frac{b}{(s^2+b^2)^2}$
10.	$\frac{1}{2b}t\sin bt$	$\to$	$\frac{s}{(s^2+b^2)^2}$

Laplace Transform Principles

Linearity	$\mathcal{L}\{af(t)+bg(t)\}$	$=$	$a\mathcal{L}\{f\}+b\mathcal{L}\{g\}$
Input Derivative Principles	$\mathcal{L}\{f'(t)\}(s)$	$=$	$s\mathcal{L}\{f(t)\}-f(0)$
	$\mathcal{L}\{f''(t)\}(s)$	$=$	$s^2\mathcal{L}\{f(t)\}-sf(0)-f'(0)$
First Translation Principle	$\mathcal{L}\{e^{at}f(t)\}$	$=$	$F(s-a)$
Transform Derivative Principle	$\mathcal{L}\{-tf(t)\}(s)$	$=$	$\frac{d}{ds}F(s)$
The Dilation Principle	$\mathcal{L}\{f(bt)\}(s)$	$=$	$\frac{1}{b}\mathcal{L}\{f(t)\}(s/b)$
The Convolution Principle	$\mathcal{L}\{(f*g)(t)\}(s)$	$=$	$F(s)G(s).$

Table of Convolutions

	$f(t)$	$g(t)$	$(f * g)(t)$
1.	1	$g(t)$	$\int_0^t g(\tau)\,d\tau$
2.	t^m	t^n	$\frac{m!n!}{(m+n+1)!}t^{m+n+1}$
3.	t	$\sin at$	$\frac{at - \sin at}{a^2}$
4.	t^2	$\sin at$	$\frac{2}{a^3}(\cos at - (1 - \frac{a^2t^2}{2}))$
5.	t	$\cos at$	$\frac{1 - \cos at}{a^2}$
6.	t^2	$\cos at$	$\frac{2}{a^3}(at - \sin at)$
7.	t	e^{at}	$\frac{e^{at} - (1 + at)}{a^2}$
8.	t^2	e^{at}	$\frac{2}{a^3}(e^{at} - (a + at + \frac{a^2t^2}{2}))$
9.	e^{at}	e^{bt}	$\frac{1}{b-a}(e^{bt} - e^{at}) \quad a \neq b$
10.	e^{at}	e^{at}	te^{at}
11.	e^{at}	$\sin bt$	$\frac{1}{a^2+b^2}(be^{at} - b\cos bt - a\sin bt)$
12.	e^{at}	$\cos bt$	$\frac{1}{a^2+b^2}(ae^{at} - a\cos bt + b\sin bt)$
13.	$\sin at$	$\sin bt$	$\frac{1}{b^2-a^2}(b\sin at - a\sin bt) \quad a \neq b$
14.	$\sin at$	$\sin at$	$\frac{1}{2a}(\sin at - at\cos at)$
15.	$\sin at$	$\cos bt$	$\frac{1}{b^2-a^2}(a\cos at - a\cos bt) \quad a \neq b$
16.	$\sin at$	$\cos at$	$\frac{1}{2}t\sin at$
17.	$\cos at$	$\cos bt$	$\frac{1}{a^2-b^2}(a\sin at - b\sin bt) \quad a \neq b$
18.	$\cos at$	$\cos at$	$\frac{1}{2a}(at\cos at + \sin at)$

Partial Fraction Expansion Theorems

The following two theorems are the main partial fractions expansion theorems, as presented in the text.

Theorem 1 (Linear Case). *Suppose a proper rational function can be written in the form*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)}$$

and $q(\lambda) \neq 0$. *Then there is a unique number* A_1 *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)} = \frac{A_1}{(s-\lambda)^n} + \frac{p_1(s)}{(s-\lambda)^{n-1} q(s)}. \tag{1}$$

The number A_1 *and the polynomial* $p_1(s)$ *are given by*

$$A_1 = \frac{p_0(\lambda)}{q(\lambda)} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - A_1 q(s)}{s-\lambda}. \tag{2}$$

Theorem 2 (Irreducible Quadratic Case). *Suppose a real proper rational function can be written in the form*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)},$$

where s^2+cs+d *is an irreducible quadratic that is factored completely out of* $q(s)$. *Then there is a unique linear term* $B_1 s + C_1$ *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)} = \frac{B_1 s + C_1}{(s^2+cs+d)^n} + \frac{p_1(s)}{(s^s+cs+d)^{n-1} q(s)}. \tag{3}$$

If $a+ib$ *is a complex root of* s^2+cs+d *then* $B_1 s + C_1$ *and the polynomial* $p_1(s)$ *are given by*

$$B_1(a+ib) + C_1 = \frac{p_0(a+ib)}{q(a+ib)} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - (B_1 s + C_1) q(s)}{s^2+cs+d}. \tag{4}$$