Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* A table of Laplace transforms and a convolution table have been appended to the exam.

1. Let $A = \begin{bmatrix} 1 & 2 \\ -3 & -4 \end{bmatrix}$.

   (a) [**3 Points**] Verify that the characteristic polynomial of A is $c_A(s) = (s+2)(s+1)$.

   ▶ **Solution.**

$$\begin{aligned} c_A(s) &= \det(sI - A) = \det \begin{bmatrix} s-1 & -2 \\ 3 & s+4 \end{bmatrix} = (s-1)(s+4) - (-2)3 \\ &= s^2 + 3s - 4 + 6 = s^2 + 3s + 2 = (s+1)(s+2). \end{aligned}$$

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   (b) [**10 Points**] Compute the matrix exponential e^{At}.

   ▶ **Solution.** Use Fulmer's method. The roots of $c_A(s)$ are -1 and -2 so $\mathcal{B}_{c_A(s)} = \{e^{-t}, e^{-2t}\}$. Thus, $e^{At} = M_1 e^{-t} + M_2 e^{-2t}$ for constant matrices M_1 and M_2. Differentiation gives $Ae^{At} = -M_1 e^{-t} - 2M_2 e^{-2t}$, and evaluating both equations at $t = 0$ gives

$$\begin{aligned} I &= M_1 + M_2 \\ A &= -M_1 - 2M_2 \end{aligned}$$

   Solving for M_1 and M_2 gives

$$\begin{aligned} M_1 &= A + 2I = \begin{bmatrix} 3 & 2 \\ -3 & -2 \end{bmatrix} \\ M_2 &= -(I + A) = \begin{bmatrix} -2 & -2 \\ 3 & 3 \end{bmatrix}. \end{aligned}$$

   Thus,

$$\begin{aligned} e^{At} &= \begin{bmatrix} 3 & 2 \\ -3 & -2 \end{bmatrix} e^{-t} + \begin{bmatrix} -2 & -2 \\ 3 & 3 \end{bmatrix} e^{-2t} \\ &= \begin{bmatrix} 3e^{-t} - 2e^{-2t} & 2e^{-t} - 2e^{-2t} \\ -3e^{-t} + 3e^{-2t} & -2e^{-t} + 3e^{-2t} \end{bmatrix}. \end{aligned}$$

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   (c) [**7 Points**] Solve the initial value problem $\mathbf{y}' = A\mathbf{y}$, $\mathbf{y}(0) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

▶ **Solution.**

$$\mathbf{y}(t) = e^{At}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 3e^{-t} - 2e^{-2t} & 2e^{-t} - 2e^{-2t} \\ -3e^{-t} + 3e^{-2t} & -2e^{-t} + 3e^{-2t} \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2e^{-2t} \\ -3e^{-2t} \end{bmatrix}.$$

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(d) [**10 Points**] Solve the initial value problem $\mathbf{y}' = A\mathbf{y} + \mathbf{f}(t)$, $\mathbf{y}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, where $\mathbf{f}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

▶ **Solution.**

$$\begin{aligned} \mathbf{y}(t) &= e^{At}\begin{bmatrix} 0 \\ 0 \end{bmatrix} + e^{At} * \mathbf{f}(t) \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 3e^{-t} - 2e^{-2t} & 2e^{-t} - 2e^{-2t} \\ -3e^{-t} + 3e^{-2t} & -2e^{-t} + 3e^{-2t} \end{bmatrix} * \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} (2e^{-t} - 2e^{-2t}) * 1 \\ (-2e^{-t} + 3e^{-2t}) * 1 \end{bmatrix} \\ &= \begin{bmatrix} \int_0^t (2e^{-u} - 2e^{-2u})\,du \\ \int_0^t (-2e^{-u} + 3e^{-2u})\,du \end{bmatrix} \\ &= \begin{bmatrix} -2e^{-t} + e^{-2t} + 1 \\ 2e^{-t} - \frac{3}{2}e^{-2t} - \frac{1}{2} \end{bmatrix}. \end{aligned}$$

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2. Consider the following first order linear system of differential equations

$$\begin{aligned} y_1' &= -2y_1 + y_2 \\ y_2' &= -4y_1 + 2y_2 \end{aligned}$$

(a) [**5 Points**] Write the system in matrix form $\mathbf{y}' = A\mathbf{y}$.

▶ **Solution.**

$$\mathbf{y}' = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix}\mathbf{y}.$$

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(b) [**15 Points**] Solve this system with the initial conditions $y_1(0) = 5$, $y_2(0) = -3$.

▶ **Solution.** First compute e^{At}: $sI - A = \begin{bmatrix} s+2 & -1 \\ 4 & s-2 \end{bmatrix}$, so $c_A(s) = (s+2)(s-2) + 4 = s^2$ and

$$(sI - A)^{-1} = \frac{1}{s^2}\begin{bmatrix} s-2 & 1 \\ -4 & s+2 \end{bmatrix} = \begin{bmatrix} \frac{1}{s} - \frac{2}{s^2} & \frac{1}{s^2} \\ \frac{-4}{s^2} & \frac{1}{s} + \frac{2}{s^2} \end{bmatrix}.$$

Thus,

$$e^{At} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\} = \begin{bmatrix} 1-2t & t \\ -4t & 1+2t \end{bmatrix},$$

and

$$\begin{aligned} \mathbf{y}(t) &= e^{At}\begin{bmatrix} 5 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} 1-2t & t \\ -4t & 1+2t \end{bmatrix}\begin{bmatrix} 5 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} 5-13t \\ -3-26t \end{bmatrix}. \end{aligned}$$

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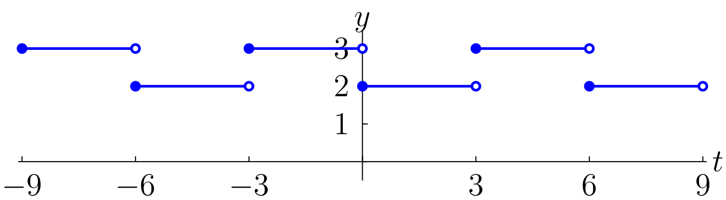
3. Let $f(t)$ be the square wave function of period 6 that is defined on the interval $[0, 6)$ by

$$f(t) = \begin{cases} 2 & \text{if } 0 \le t < 3, \\ 3 & \text{if } 3 \le t < 6. \end{cases}$$

(a) [**4 Points**] Sketch the graph of $f(t)$ on the interval $[-9, 9]$.

▶ **Solution.**



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(b) [**3 Points**] Determine the points where $f(t)$ is discontinuous.

▶ **Solution.** $f(t)$ is discontinuous for $0, \pm 3, \pm 6, \pm 9, \cdots$. That is, for all $t = 3n$ for n an integer. ◀

(c) [**3 Points**] Fill in the blanks to complete the convergence theorem for Fourier series:

If $f(t)$ is periodic and piecewise $\boxed{\textbf{smooth}}$, *then its Fourier series converges to*

i. $\boxed{f(t)}$ *at each point t where $f(t)$ is continuous.*

ii. $\boxed{(f(t^+) + f(t^-))/2}$ *at each point t where $f(t)$ is not continuous.*

(d) [**16 Points**] Compute the Fourier series of $f(t)$.

▶ **Solution.**

$$a_0 = \frac{1}{3}\int_{-3}^{3} f(t)\,dt = \frac{1}{3}\int_{-3}^{0} 3\,dt + \frac{1}{3}\int_{0}^{3} 2\,dt = 3 + 2 = 5.$$

For $n \geq 1$,

$$\begin{aligned}
a_n &= \frac{1}{3}\int_{-3}^{3} f(t)\cos\frac{n\pi}{3}t\,dt \\
&= \frac{1}{3}\int_{-3}^{0} 3\cos\frac{n\pi}{3}t\,dt + \frac{1}{3}\int_{0}^{3} 2\cos\frac{n\pi}{3}t\,dt \\
&= \frac{3}{n\pi}\sin\frac{n\pi}{3}t\Big|_{-3}^{0} + \frac{2}{n\pi}\sin\frac{n\pi}{3}t\Big|_{0}^{3} \\
&= \frac{3}{n\pi}(\sin 0 - \sin(-n\pi)) + \frac{2}{n\pi}(\sin n\pi - \sin 0) \\
&= 0 \quad \text{since } \sin n\pi = 0.
\end{aligned}$$

For $n \geq 1$,

$$\begin{aligned}
b_n &= \frac{1}{3}\int_{-3}^{3} f(t)\sin\frac{n\pi}{3}t\,dt \\
&= \frac{1}{3}\int_{-3}^{0} 3\sin\frac{n\pi}{3}t\,dt + \frac{1}{3}\int_{0}^{3} 2\sin\frac{n\pi}{3}t\,dt \\
&= -\frac{3}{n\pi}\cos\frac{n\pi}{3}t\Big|_{-3}^{0} - \frac{2}{n\pi}\cos\frac{n\pi}{3}t\Big|_{0}^{3} \\
&= -\frac{3}{n\pi}(1-\cos n\pi) - \frac{2}{n\pi}(\cos n\pi - 1) \\
&= -\frac{1}{n\pi}(1-\cos n\pi) = \begin{cases} 0 & n \text{ is even} \\ -\frac{2}{n\pi} & n \text{ is odd.} \end{cases}
\end{aligned}$$

Thus, the Fourier series of $f(t)$ is

$$f(t) \sim \frac{5}{2} + \sum_{n=\text{odd}} -\frac{2}{n\pi}\sin\frac{n\pi}{3}t.$$

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(e) **[4 Points]** Let $g(t)$ denote the sum of the Fourier series found in part (d). Compute $g(0)$, $g(1)$, and $g(4)$ and $g(-9)$.

▶ **Solution.** $g(0) = (f(0^+) + f(0^-))/2 = (2+3)/2 = 5/2$ since f has a jump discontinuity at 0, $g(1) = f(1) = 2$ since f is continuous at 1, $g(4) = f(4) = 3$, $g(-9) = (f(-9^+) + f(-9^-))/2 = (3+2)/2 = 5/2$. ◀

4. Consider the function

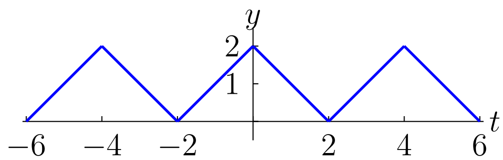
$$f(t) = 2 - t, \qquad 0 < t < 2.$$

(a) **[4 Points]** Let $f_e(t)$ be the even periodic extension of $f(t)$. Give an explicit formula for $f_e(t)$ and draw the graph of $f_3(t)$ on the interval $[-6, 6]$.

▶ **Solution.** The even periodic extension is

$$f_e(t) = \begin{cases} f(t) & \text{if } 0 < t < 2 \\ f(-t) & \text{if } -2 < t < 0 \end{cases} = \begin{cases} 2-t & \text{if } 0 < t < 2 \\ 2+t & \text{if } -2 < t < 0 \end{cases}, \quad f(t+4) = f(t).$$

The graph is:



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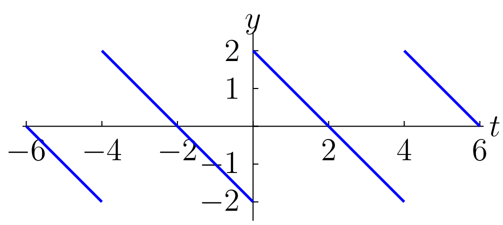
(b) [**4 Points**] Let $f_o(t)$ be the odd periodic extension of $f(t)$. Give an explicit formula for $f_o(t)$ and draw the graph of $f_3(t)$ on the interval $[-6, 6]$.
The following integration formulas may be of use:

$$\int t \sin at\, dt = \frac{1}{a^2}\sin at - \frac{t}{a}\cos at \qquad \int t\cos at\, dt = \frac{1}{a^2}\cos at + \frac{t}{a}\sin at.$$

▶ **Solution.** The odd periodic extension is

$$f_0(t) = \begin{cases} f(t) & \text{if } 0 < t < 2 \\ -f(-t) & \text{if } -2 < t < 0 \end{cases} = \begin{cases} 2-t & \text{if } 0 < t < 2 \\ -2-t & \text{if } -2 < t < 0 \end{cases}, \quad f(t+4) = f(t).$$

The graph is:



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(c) [**12 Points**] Compute the Fourier sine series of $f(t)$.

▶ **Solution.** The sine series has only sine terms. For $n \geq 1$,

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(t)\sin\frac{n\pi}{L}t\,dt \\ &= \frac{2}{2}\int_0^2 (2-t)\sin\frac{n\pi}{2}t\,dt \\ &= (2-t)\frac{-2}{n\pi}\cos\frac{n\pi}{2}t\Big|_0^2 - \int_0^2 \frac{-2}{n\pi}\cos\frac{n\pi}{2}t\,(-dt) \\ &= \frac{4}{n\pi} - \left(\frac{2}{n\pi}\right)^2 \sin\frac{n\pi}{2}t\Big|_0^2 \\ &= \frac{4}{n\pi}. \end{aligned}$$

Thus, the Fourier sine series is

$$f(t) \sim \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin \frac{n\pi}{2} t.$$

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Laplace Transform Table

	$f(t)$	$\longleftrightarrow$	$F(s) = \mathcal{L}\left\{f(t)\right\}(s)$
1.	1	$\longleftrightarrow$	$\dfrac{1}{s}$
2.	t^n	$\longleftrightarrow$	$\dfrac{n!}{s^{n+1}}$
3.	e^{at}	$\longleftrightarrow$	$\dfrac{1}{s-a}$
4.	$t^n e^{at}$	$\longleftrightarrow$	$\dfrac{n!}{(s-a)^{n+1}}$
5.	$\cos bt$	$\longleftrightarrow$	$\dfrac{s}{s^2+b^2}$
6.	$\sin bt$	$\longleftrightarrow$	$\dfrac{b}{s^2+b^2}$
7.	$e^{at}\cos bt$	$\longleftrightarrow$	$\dfrac{s-a}{(s-a)^2+b^2}$
8.	$e^{at}\sin bt$	$\longleftrightarrow$	$\dfrac{b}{(s-a)^2+b^2}$
9.	$h(t-c)$	$\longleftrightarrow$	$\dfrac{e^{-sc}}{s}$
10.	$\delta_c(t) = \delta(t-c)$	$\longleftrightarrow$	e^{-sc}

Laplace Transform Principles

Linearity	$\mathcal{L}\left\{af(t)+bg(t)\right\}$	$=$	$a\mathcal{L}\left\{f\right\}+b\mathcal{L}\left\{g\right\}$
Input Derivative Principles	$\mathcal{L}\left\{f'(t)\right\}(s)$	$=$	$s\mathcal{L}\left\{f(t)\right\}-f(0)$
	$\mathcal{L}\left\{f''(t)\right\}(s)$	$=$	$s^2\mathcal{L}\left\{f(t)\right\}-sf(0)-f'(0)$
First Translation Principle	$\mathcal{L}\left\{e^{at}f(t)\right\}$	$=$	$F(s-a)$
Transform Derivative Principle	$\mathcal{L}\left\{-tf(t)\right\}(s)$	$=$	$\dfrac{d}{ds}F(s)$
Second Translation Principle	$\mathcal{L}\left\{h(t-c)f(t-c)\right\}$	$=$	$e^{-sc}F(s)$, or
	$\mathcal{L}\left\{g(t)h(t-c)\right\}$	$=$	$e^{-sc}\mathcal{L}\left\{g(t+c)\right\}$.
The Convolution Principle	$\mathcal{L}\left\{(f*g)(t)\right\}(s)$	$=$	$F(s)G(s)$.

Table of Convolutions

	$f(t)$	$g(t)$	$(f * g)(t)$
1.	1	$g(t)$	$\int_0^t g(\tau)\,d\tau$
2.	t^m	t^n	$\frac{m!n!}{(m+n+1)!}t^{m+n+1}$
3.	t	$\sin at$	$\frac{at - \sin at}{a^2}$
4.	t^2	$\sin at$	$\frac{2}{a^3}(\cos at - (1 - \frac{a^2t^2}{2}))$
5.	t	$\cos at$	$\frac{1 - \cos at}{a^2}$
6.	t^2	$\cos at$	$\frac{2}{a^3}(at - \sin at)$
7.	t	e^{at}	$\frac{e^{at} - (1 + at)}{a^2}$
8.	t^2	e^{at}	$\frac{2}{a^3}(e^{at} - (a + at + \frac{a^2t^2}{2}))$
9.	e^{at}	e^{bt}	$\frac{1}{b-a}(e^{bt} - e^{at}) \quad a \neq b$
10.	e^{at}	e^{at}	te^{at}
11.	e^{at}	$\sin bt$	$\frac{1}{a^2+b^2}(be^{at} - b\cos bt - a\sin bt)$
12.	e^{at}	$\cos bt$	$\frac{1}{a^2+b^2}(ae^{at} - a\cos bt + b\sin bt)$
13.	$\sin at$	$\sin bt$	$\frac{1}{b^2-a^2}(b\sin at - a\sin bt) \quad a \neq b$
14.	$\sin at$	$\sin at$	$\frac{1}{2a}(\sin at - at\cos at)$
15.	$\sin at$	$\cos bt$	$\frac{1}{b^2-a^2}(a\cos at - a\cos bt) \quad a \neq b$
16.	$\sin at$	$\cos at$	$\frac{1}{2}t\sin at$
17.	$\cos at$	$\cos bt$	$\frac{1}{a^2-b^2}(a\sin at - b\sin bt) \quad a \neq b$
18.	$\cos at$	$\cos at$	$\frac{1}{2a}(at\cos at + \sin at)$