Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* A table of Laplace transforms and the statement of the partial fraction decomposition theorems are attached to the exam.

In Exercises 1 – 7, solve the given differential equation. If initial values are given, solve the initial value problem. Otherwise, give the general solution. Some problems may be solvable by more than one technique. You are free to choose whatever technique that you deem to be most appropriate.

1. [**12 Points**] $y' - 2y = 4te^{2t} + 6, \quad y(0) = 2.$
2. [**12 Points**] $2tyy' = 1 + y^2, \quad y(2) = 3.$
3. [**12 Points**] $2y'' + 5y' + 2y = 0, \quad y(0) = 1,\ y'(0) = 1.$
4. [**12 Points**] $4y'' + 9y = 0, \quad y(0) = -1,\ y'(0) = 6.$
5. [**12 Points**] $2t^2y'' + 5ty' - 2y = 0.$
6. [**12 Points**] $y'' + 25y = 2\delta(t - \pi), \quad y(0) = 2,\ y'(0) = 3.$ Recall that $\delta(t - c))$ refers to the Dirac delta function providing a unit impulse at time c.
7. [**12 Points**] $y'' + 5y' + 4y = 6\sin 2t.$
8. [**12 Points**] Find a particular solution for $t > 0$ of the differential equation

$$y'' + \frac{1}{t}y' + \frac{1}{t^2}y = 72t^3,$$

given that the general solution of the associated homogeneous equation is

$$y_h(t) = c_1 t + c_2 t^{-1}.$$

9. [**10 Points**] Compute the Laplace transform $F(s)$ of the function $f(t)$ defined as follows:

$$f(t) = (t^2 - 2t + 1)h(t - 3).$$

10. [**10 Points**] Compute the inverse Laplace transform of the following function:

$$F(s) = e^{-2s}\frac{4}{(s^2 + 4s + 3)(s + 1)}$$

11. [**14 Points**] Solve the following system of differential equations

$$\begin{aligned} y_1' &= 3y_1 - y_2 & \qquad y_1(0) &= 3 \\ y_2' &= -y_1 + 3y_2 & \qquad y_2(0) &= 2. \end{aligned}$$

12. [**10 Points**]Let $f(t) = t^2 + 1$, for $0 < t < 2$.

    (a) Let $g_1(t)$ be the **odd** periodic extension of $f(t)$ of period $P = 4$. Sketch 3 periods of $g_1(t)$ on the interval $-6 < t < 6$.

    (b) To what value does the Fourier series of $g_1(t)$ converge at $t = -1$? At $t = 4$?

    (c) Let $g_2(t)$ be the **even** periodic extension of $f(t)$ of period $P = 4$. Sketch 3 periods of $g_2(t)$ on the interval $-6 < t < 6$.

    (d) Find the constant term $\frac{a_0}{2}$ of the Fourier series of the even periodic function $g_2(t)$.

    (e) State TRUE/FALSE with reason. For the even periodic function $g_2(t)$ in part (c), the Fourier cosine coefficients a_n, $n \geq 1$, are given by

$$a_n = \int_0^2 (t^2 + 1) \sin \frac{n\pi}{2} t \, dt.$$

13. [**10 Points**] A 10-gallon tank initially contains 4 gallons of pure water. Starting at 1 PM, brine with a salt concentration of 1/4 pound of salt per gallon runs into the tank at the rate of 2 gallons per hour. Also starting at 1 PM, the well-mixed solution is drained from the bottom of the tank at the rate of 1 gallon per hour.

    (a) At what time will the tank begin to overflow.

    (b) Set up a differential equation with initial value that will describe the amount of salt in the tank at any time until it overflows. Be sure to identify the variables that you are using. Just write down the differential equation; you do **not** need to solve it.

Laplace Transform Table

	$f(t)$	$\to$	$F(s) = \mathcal{L}\{f(t)\}(s)$
1.	1	$\to$	$\dfrac{1}{s}$
2.	t^n	$\to$	$\dfrac{n!}{s^{n+1}}$
3.	e^{at}	$\to$	$\dfrac{1}{s-a}$
4.	$t^n e^{at}$	$\to$	$\dfrac{n!}{(s-a)^{n+1}}$
5.	$\cos bt$	$\to$	$\dfrac{s}{s^2+b^2}$
6.	$\sin bt$	$\to$	$\dfrac{b}{s^2+b^2}$
7.	$e^{at}\cos bt$	$\to$	$\dfrac{s-a}{(s-a)^2+b^2}$
8.	$e^{at}\sin bt$	$\to$	$\dfrac{b}{(s-a)^2+b^2}$
9.	$h(t-c)$	$\to$	$\dfrac{e^{-sc}}{s}$
10.	$\delta(t-c)$	$\to$	e^{-sc}

Laplace Transform Principles

Linearity	$\mathcal{L}\{af(t)+bg(t)\}$	$=$	$a\mathcal{L}\{f\}+b\mathcal{L}\{g\}$
Input Derivative Principles	$\mathcal{L}\{f'(t)\}(s)$	$=$	$s\mathcal{L}\{f(t)\}-f(0)$
	$\mathcal{L}\{f''(t)\}(s)$	$=$	$s^2\mathcal{L}\{f(t)\}-sf(0)-f'(0)$
First Translation Principle	$\mathcal{L}\{e^{at}f(t)\}$	$=$	$F(s-a)$
Transform Derivative Principle	$\mathcal{L}\{-tf(t)\}(s)$	$=$	$\dfrac{d}{ds}F(s)$
Second Translation Principle	$\mathcal{L}\{h(t-c)f(t-c)\}$	$=$	$e^{-sc}F(s)$, or
	$\mathcal{L}\{g(t)h(t-c)\}$	$=$	$e^{-sc}\mathcal{L}\{g(t+c)\}$.
The Convolution Principle	$\mathcal{L}\{(f*g)(t)\}(s)$	$=$	$F(s)G(s)$.

Partial Fraction Expansion Theorems

The following two theorems are the main partial fractions expansion theorems, as presented in the text.

Theorem 1 (Linear Case). *Suppose a proper rational function can be written in the form*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)}$$

and $q(\lambda) \neq 0$. Then there is a unique number A_1 and a unique polynomial $p_1(s)$ such that

$$\frac{p_0(s)}{(s-\lambda)^n q(s)} = \frac{A_1}{(s-\lambda)^n} + \frac{p_1(s)}{(s-\lambda)^{n-1} q(s)}. \tag{1}$$

The number A_1 and the polynomial $p_1(s)$ are given by

$$A_1 = \frac{p_0(\lambda)}{q(\lambda)} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - A_1 q(s)}{s-\lambda}. \tag{2}$$

Theorem 2 (Irreducible Quadratic Case). *Suppose a real proper rational function can be written in the form*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)},$$

where s^2+cs+d is an irreducible quadratic that is factored completely out of $q(s)$. Then there is a unique linear term $B_1 s + C_1$ and a unique polynomial $p_1(s)$ such that

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)} = \frac{B_1 s + C_1}{(s^2+cs+d)^n} + \frac{p_1(s)}{(s^s+cs+d)^{n-1} q(s)}. \tag{3}$$

If $a+ib$ is a complex root of s^2+cs+d then $B_1 s + C_1$ and the polynomial $p_1(s)$ are given by

$$B_1(a+ib) + C_1 = \frac{p_0(a+ib)}{q(a+ib)} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - (B_1 s + C_1) q(s)}{s^2+cs+d}. \tag{4}$$