Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* A table of Laplace transforms and the statement of the partial fraction decomposition theorems are attached to the exam.

In Exercises 1 – 7, solve the given differential equation. If initial values are given, solve the initial value problem. Otherwise, give the general solution. Some problems may be solvable by more than one technique. You are free to choose whatever technique that you deem to be most appropriate.

1. [**12 Points**] $y' - 2y = 4te^{2t} + 6, \quad y(0) = 2.$

   ▶ **Solution.** This is a first order linear equation, so compute an integrating factor

   $$\mu(t) = e^{-\int 2\,dt} = e^{-2t},$$

   and multiply the equation by $\mu(t) = e^{-2t}$ to get $e^{-2t}y' - 2e^{-2t}y = 4t + 6e^{-2t}$. The left hand side is $(e^{-2t}y)'$ so we get the equation $(e^{-2t}y)' = 4t + 6e^{-2t}$ and integration then gives $e^{-2t}y = 2t^2 - 3e^{-2t} + C$ so that $y(t) = 2t^2e^{2t} - 3 + Ce^{2t}$. Using the initial condition $y(0) = 2$ gives $2 = y(0) = -3 + C$, so that $C = 5$. Hence,

   $$\boxed{y(t) = 2t^2e^{2t} - 3 + 5e^{2t}.}$$

   ◀

2. [**12 Points**] $2tyy' = 1 + y^2, \quad y(2) = 3.$

   ▶ **Solution.** This equation is separable. Separate the variables to get $\dfrac{2y}{1+y^2}y' = \dfrac{1}{t}$. Write in differential form and integrate to get:

   $$\int \frac{2y}{1+y^2}\,dy = \int \frac{dt}{t}.$$

   Integrating gives $\ln(1 + y^2) = \ln t + C$, and taking the exponential of both sides gives $1 + y^2 = Bt$ where $B = e^C$ is a constant. Thus, $y^2 = Bt - 1$ and using the initial condition $y = 3$ when $t = 2$ gives $9 = 2B - 1$ so that $B = 5$. Hence

   $$\boxed{y(t) = \sqrt{5t - 1}.}$$

   ◀

3. [**12 Points**] $2y'' + 5y' + 2y = 0, \quad y(0) = 1,\ y'(0) = 1.$

▶ **Solution.** This equation has characteristic polynomial

$$q(s) = 2s^2 + 5s + 2 = (2s+1)(s+2)$$

which has roots $-1/2$ and -2. Hence the general solution of the homogeneous equation is $y = c_1 e^{-t/2} + c_2 e^{-2t}$. The initial conditions mean that c_1 and c_2 satisfy

$$\begin{aligned} c_1 + c_2 &= 1 \\ (-1/2)c_1 - 2c_2 &= 1. \end{aligned}$$

Solve these equations to get $c_1 = 2$, $c_2 = -1$. Hence,

$$\boxed{y = 2e^{-t/2} - e^{-2t}.}$$

◀

4. [**12 Points**] $4y'' + 9y = 0, \quad y(0) = -1,\ y'(0) = 6.$

▶ **Solution.** This is a constant coefficient homogeneous linear equation with characteristic polynomial $q(s) = 4s^2 + 9$ which has roots $\pm\frac{3}{2}i$. Hence, the general solution is $y = c_1 \cos(3/2)t + c_2 \sin(3/2)t$. The initial conditions imply that $-1 = y(0) = c_1$ and $6 = y'(0) = (3/2)c_2$ so that $c_2 = 4$. Thus

$$\boxed{y = -\cos\frac{3t}{2} + 4\sin\frac{3t}{2}.}$$

◀

5. [**12 Points**] $2t^2y'' + 5ty' - 2y = 0.$

▶ **Solution.** This is a Cauchy-Euler equation with indicial polynomial

$$q(s) = 2s(s-1) + 5s - 2 = 2s^3 + 3s - 2 = (2s-1)(s+2),$$

which has roots $1/2$ and -2. Thus, the general solution is

$$\boxed{y(t) = c_1 t^{1/2} + c_2 t^{-2}.}$$

◀

6. [**12 Points**] $y'' + 25y = 2\delta(t-\pi), \quad y(0) = 2,\ y'(0) = 3.$ Recall that $\delta(t-c))$ refers to the Dirac delta function providing a unit impulse at time c.

▶ **Solution.** Use the Laplace transform method. Let $Y(s) = \mathcal{L}\{y(t)\}$ where $y(t)$ is the unknown solution of the initial value problem. Applying the Laplace transform to the differential equation gives:

$$s^2Y(s) - 2s - 3 + 25Y(s) = 2e^{-\pi s}.$$

Solve for $Y(s)$:

$$Y(s) = \frac{2s+3}{s^2+25} + \frac{2}{s^2+25}e^{-\pi s}.$$

Then take the inverse laplace transform to get:

$$\boxed{y(t) = 2\cos 5t + \frac{3}{5}\sin 5t + \frac{2}{5}h(t-\pi)\sin 5(t-\pi).}$$

◀

7. [**12 Points**] $y'' + 5y' + 4y = 6\sin 2t$.

▶ **Solution.** Use the method of undetermined coefficients. The characteristic polynomial is $q(s) = s^2 + 5s + 4 = (s+4)(s+1)$ which has roots -4 and -1. Thus $\mathcal{B}_q = \{e^{-4t},\, e^{-t}\}$ and $y_h = c_1e^{-4t} + c_2e^{-t}$. Since $\mathcal{L}\{6\sin 2t\} = \dfrac{12}{s^2+4}$ the denominator is $v = s^2 + 4$ and $qv = (s+4)(s+1)(s^2+4)$. Hence,

$$\mathcal{B}_{qv} \setminus \mathcal{B}_q = \left\{e^{-4t},\, e^{-t},\, \cos 2t,\, \sin 2t\right\} \setminus \left\{e^{-4t},\, e^{-t}\right\} = \{\cos 2t,\, \sin 2t\}.$$

Therefore, the test function for y_p is $y_p = A\cos 2t + B\sin 2t$. Compute the derivatives:

$$\begin{aligned} y_p' &= -2A\sin 2t + 2B\cos 2t \\ y_p'' &= -4A\cos 2t - 4B\sin 2t. \end{aligned}$$

Substituting into the differential equation gives

$$\begin{aligned} 6\sin 2t &= y_p'' + 5y_p' + y_p \\ &= (-4A\cos 2t - 4B\sin 2t) + 5(-2A\sin 2t + 2B\cos 2t) + 4(A\cos 2t + B\sin 2t) \\ &= 10B\cos 2t + -10A\sin 2t. \end{aligned}$$

Comparing the coefficients of $\cos 2t$ and $\sin 2t$ on both sides of this equation shows that A and B satisfy the system of linear equations

$$\begin{aligned} 10B &= 0 \\ -10A &= 6. \end{aligned}$$

Thus $B = 0$ and $A = -5/3$. Thus,

$$y_p = -\frac{5}{3}\cos 2t,$$

and

$$y_g = y_h + y_p = c_1 e^{-4t} + c_2 t e^{-t} - \frac{5}{3}\cos 2t.$$

◀

8. [**12 Points**] Find a particular solution for $t > 0$ of the differential equation

$$y'' + \frac{1}{t}y' - \frac{1}{t^2}y = 72t^3,$$

given that the general solution of the associated homogeneous equation is

$$y_h(t) = c_1 t + c_2 t^{-1}.$$

▶ **Solution.** Use variation of parameters. A particular solution is given by

$$y_p = u_1 t + u_2 t^{-1}$$

where u_1' and u_2' satisfy the equations:

$$\begin{aligned} u_1' t + u_2' t^{-1} &= 0 \\ u_1' - u_2' t^{-2} &= 72t^3. \end{aligned}$$

Solving for u_1' and u_2' via Cramer's rule gives

$$u_1' = \frac{\begin{vmatrix} 0 & t^{-1} \\ 72t^3 & -t^{-2} \end{vmatrix}}{\begin{vmatrix} t & t^{-1} \\ 1 & -t^{-2} \end{vmatrix}} = \frac{-72t^2}{-2t^{-1}} = 36t^3,$$

and

$$u_2' = \frac{\begin{vmatrix} t & 0 \\ 1 & 72t^3 \end{vmatrix}}{\begin{vmatrix} t & t^{-1} \\ 1 & -t^{-2} \end{vmatrix}} = \frac{72t^4}{-2t^{-1}} = -36t^5$$

Integrating then gives

$$u_1 = 9t^4 \quad \text{and} \quad u_2 = -6t^6,$$

which gives

$$y_p = 9t^4 \cdot t - 6t^6 \cdot t^{-1} = 3t^5.$$

◀

9. [**10 Points**] Compute the Laplace transform $F(s)$ of the function $f(t)$ defined as follows:

$$f(t) = (t^2 - 2t + 1)h(t - 3).$$

▶ **Solution.** $f(t) = (t^2 - 2t + 1)h(t-3) = (t-1)^2 h(t-3)$. By the second translation theorem

$$\begin{aligned} F(s) &= \mathcal{L}\left\{f(t)\right\} = \mathcal{L}\left\{(t-1)^2 h(t-3)\right\} \\ &= e^{-3s}\mathcal{L}\left\{((t+3)-1)^2\right\} = e^{-3s}\mathcal{L}\left\{(t+2)^2\right\} \\ &= e^{-3s}\mathcal{L}\left\{t^2 + 4t + 4\right\} \\ &= e^{-3s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right). \end{aligned}$$

◀

10. [**10 Points**] Compute the inverse Laplace transform of the following function:

$$F(s) = e^{-2s}\frac{4}{(s^2+4s+3)(s+1)}$$

▶ **Solution.** Let $G(s) = \dfrac{4}{(s^2+4s+3)(s+1)}$. Then from the second translation theorem

$$f(t) = \mathcal{L}^{-1}\left\{F(s)\right\} = \mathcal{L}^{-1}\left\{e^{-2s}G(s)\right\} = h(t-2)g(t-2),$$

where $g(t) = \mathcal{L}^{-1}\left\{G(s)\right\}$. Use partial fractions on $G(s)$ to compute $g(t)$. First,

$$\frac{4}{(s^2+4s+3)(s+1)} = \frac{4}{(s+3)(s+1)^2} = \frac{A}{s+3} + \frac{p_1(s)}{(s+1)^2}$$

where

$$A = \left.\frac{4}{(s+1)^2}\right|_{s=-3} = 1$$

and

$$\begin{aligned} p_1(s) &= \frac{4-(s+1)^2}{s+3} = \frac{4-(s^2+2s+1)}{s+3} \\ &= \frac{4-(s^2+2s+1)}{s+3} = \frac{-s^2-2s+3}{s+3} \\ &= \frac{-(s+3)(s-1)}{s+3} = -(s-1). \end{aligned}$$

Thus,

$$\begin{aligned} G(s) &= \frac{4}{(s^2+4s+3)(s+1)} = \frac{1}{s+3} - \frac{s-1}{(s+1)^2} \\ &= \frac{1}{s+3} - \frac{(s+1)-2}{(s+1)^2} \\ &= \frac{1}{s+3} - \frac{1}{s+1} + \frac{2}{(s+1)^2}. \end{aligned}$$

Hence, $g(t) = \mathcal{L}^{-1}\left\{G(s)\right\} = e^{-3t} - e^{-t} + 2te^{-t}$ and

$$\begin{aligned} f(t) &= h(t-2)g(t-2) \\ &= h(t-2)\left(e^{-3(t-2)} - e^{-(t-2)} + 2(t-2)e^{-(t-2)}\right). \end{aligned}$$

◀

11. [**12 Points**] Solve the following system of differential equations

$$\begin{aligned} y_1' &= 3y_1 - y_2 & \qquad y_1(0) &= 3 \\ y_2' &= -y_1 + 3y_2 & \qquad y_2(0) &= 2. \end{aligned}$$

▶ **Solution.** First write the system in matrix form

$$\mathbf{y}' = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \mathbf{y} = A\mathbf{y}.$$

Now compute e^{At}: Use Fulmer's method. Compute the characteristic polynomial $c_A(s)$:

$$\begin{aligned} c_A(s) = \det(sI - A) = \det \begin{bmatrix} s-3 & 1 \\ 1 & s-3 \end{bmatrix} &= (s-3)^2 - 1 \\ = s^2 - 6s + 9 - 1 = s^2 - 6s + 8 &= (s-4)(s-2). \end{aligned}$$

The roots of $c_A(s)$ are 2 and 4 so $\mathcal{B}_{c_A(s)} = \{e^{2t}, e^{4t}\}$. Thus, $e^{At} = M_1e^{2t} + M_2e^{4t}$ for constant matrices M_1 and M_2. Differentiation gives $Ae^{At} = 2M_1e^{2t} + 4M_2e^{4t}$, and evaluating both equations at $t = 0$ gives

$$\begin{aligned} I &= M_1 + M_2 \\ A &= 2M_1 + 4M_2 \end{aligned}$$

Solving for M_1 and M_2 gives

$$\begin{aligned} M_1 &= \frac{1}{2}(4I - A) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ M_2 &= \frac{1}{2}(A - 2I) = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \end{aligned}$$

Thus,

$$\begin{aligned} e^{At} &= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} e^{2t} + \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} e^{4t} \\ &= \frac{1}{2}\begin{bmatrix} e^{2t} + e^{4t} & e^{2t} - e^{4t} \\ e^{2t} - e^{4t} & e^{2t} + e^{4t} \end{bmatrix}. \end{aligned}$$

Then the solution of the initial value problem is $\mathbf{y}(t) = e^{At}\mathbf{y}(0)$, that is

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{2t}+e^{4t} & e^{2t}-e^{4t} \\ e^{2t}-e^{4t} & e^{2t}+e^{4t} \end{bmatrix}[3//2]$$
$$= \frac{1}{2}\begin{bmatrix} 5e^{2t}+e^{4t} \\ 5e^{2t}-e^{4t} \end{bmatrix}$$

◀

12. [**10 Points**]Let $f(t) = t^2 + 1$, for $0 < t < 2$.

(a) Let $g_1(t)$ be the **odd** periodic extension of $f(t)$ of period $P = 4$. Sketch 3 periods of $g_1(t)$ on the interval $-6 < t < 6$.

(b) To what value does the Fourier series of $g_1(t)$ converge at $t = -1$? At $t = 4$?

(c) Let $g_2(t)$ be the **even** periodic extension of $f(t)$ of period $P = 4$. Sketch 3 periods of $g_2(t)$ on the interval $-6 < t < 6$.

(d) Find the constant term $\frac{a_0}{2}$ of the Fourier series of the even periodic function $g_2(t)$.

(e) State TRUE/FALSE with reason. For the even periodic function $g_2(t)$ in part (c), the Fourier cosine coefficients a_n, $n \geq 1$, are given by

$$a_n = \int_0^2 (t^2+1)\sin\frac{n\pi}{2}t\,dt.$$

13. [**10 Points**] A 10-gallon tank initially contains 4 gallons of pure water. Starting at 1 PM, brine with a salt concentration of 1/4 pound of salt per gallon runs into the tank at the rate of 2 gallons per hour. Also starting at 1 PM, the well-mixed solution is drained from the bottom of the tank at the rate of 1 gallon per hour.

(a) At what time will the tank begin to overflow.

(b) Set up a differential equation with initial value that will describe the amount of salt in the tank at any time until it overflows. Be sure to identify the variables that you are using. Just write down the differential equation; you do **not** need to solve it.

Laplace Transform Table

	$f(t)$	$\rightarrow$	$F(s) = \mathcal{L}\{f(t)\}(s)$
1.	1	$\rightarrow$	$\dfrac{1}{s}$
2.	t^n	$\rightarrow$	$\dfrac{n!}{s^{n+1}}$
3.	e^{at}	$\rightarrow$	$\dfrac{1}{s-a}$
4.	$t^n e^{at}$	$\rightarrow$	$\dfrac{n!}{(s-a)^{n+1}}$
5.	$\cos bt$	$\rightarrow$	$\dfrac{s}{s^2+b^2}$
6.	$\sin bt$	$\rightarrow$	$\dfrac{b}{s^2+b^2}$
7.	$e^{at}\cos bt$	$\rightarrow$	$\dfrac{s-a}{(s-a)^2+b^2}$
8.	$e^{at}\sin bt$	$\rightarrow$	$\dfrac{b}{(s-a)^2+b^2}$
9.	$h(t-c)$	$\rightarrow$	$\dfrac{e^{-sc}}{s}$
10.	$\delta(t-c)$	$\rightarrow$	e^{-sc}

Laplace Transform Principles

Linearity	$\mathcal{L}\{af(t)+bg(t)\}$	$=$	$a\mathcal{L}\{f\}+b\mathcal{L}\{g\}$
Input Derivative Principles	$\mathcal{L}\{f'(t)\}(s)$	$=$	$s\mathcal{L}\{f(t)\}-f(0)$
	$\mathcal{L}\{f''(t)\}(s)$	$=$	$s^2\mathcal{L}\{f(t)\}-sf(0)-f'(0)$
First Translation Principle	$\mathcal{L}\{e^{at}f(t)\}$	$=$	$F(s-a)$
Transform Derivative Principle	$\mathcal{L}\{-tf(t)\}(s)$	$=$	$\dfrac{d}{ds}F(s)$
Second Translation Principle	$\mathcal{L}\{h(t-c)f(t-c)\}$	$=$	$e^{-sc}F(s)$, or
	$\mathcal{L}\{g(t)h(t-c)\}$	$=$	$e^{-sc}\mathcal{L}\{g(t+c)\}$.
The Convolution Principle	$\mathcal{L}\{(f*g)(t)\}(s)$	$=$	$F(s)G(s)$.

Partial Fraction Expansion Theorems

The following two theorems are the main partial fractions expansion theorems, as presented in the text.

Theorem 1 (Linear Case). *Suppose a proper rational function can be written in the form*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)}$$

and $q(\lambda) \neq 0$. *Then there is a unique number* A_1 *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)} = \frac{A_1}{(s-\lambda)^n} + \frac{p_1(s)}{(s-\lambda)^{n-1} q(s)}. \tag{1}$$

The number A_1 *and the polynomial* $p_1(s)$ *are given by*

$$A_1 = \frac{p_0(\lambda)}{q(\lambda)} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - A_1 q(s)}{s-\lambda}. \tag{2}$$

Theorem 2 (Irreducible Quadratic Case). *Suppose a real proper rational function can be written in the form*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)},$$

where s^2+cs+d *is an irreducible quadratic that is factored completely out of* $q(s)$. *Then there is a unique linear term* $B_1 s + C_1$ *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)} = \frac{B_1 s + C_1}{(s^2+cs+d)^n} + \frac{p_1(s)}{(s^s+cs+d)^{n-1} q(s)}. \tag{3}$$

If $a+ib$ *is a complex root of* s^2+cs+d *then* $B_1 s + C_1$ *and the polynomial* $p_1(s)$ *are given by*

$$B_1(a+ib) + C_1 = \frac{p_0(a+ib)}{q(a+ib)} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - (B_1 s + C_1) q(s)}{s^2+cs+d}. \tag{4}$$