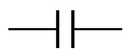
First Order Circuits

We will consider a few simple electrical circuits that lead to first order linear differential equations. These are sometimes referred to as *first order circuits*. The basic elements to be considered are:

1. Resistor 
2. Inductor 
3. Capacitor 

The current $I(t)$, expressed in units of *amperes*, through one of these elements is related to the voltage drop $V(t)$, in units of *volts*, across it as follows:

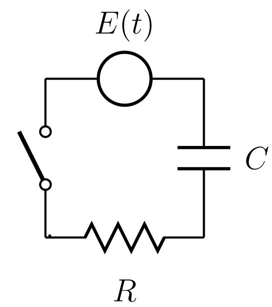
1. For a resistor, $V_R(t) = RI(t)$.
2. For an inductor, $V_L(t) = L\dfrac{dI(t)}{dt}$.
3. For a capacitor, $I(t) = C\dfrac{dV_C(t)}{dt}$, or $V_C(t) = \dfrac{1}{C}\int I(t)\,dt$.

The unit for R is the *ohm* (Ω), for L the *henry* (H), and for C the *farad* (F). We will assume that R, L, and C are constants. That is, they are independent of time t and also independent of the current I.

The analysis of closed composed of these elements and a voltage source is governed by the following two laws of Kirchhoff.

1. **Kirchoff's voltage law:** In a closed circuit the sum of the voltage drops across each element of the circuit is equal to the impressed voltage.
2. **Kirchoff's current law:** The sum of the currents flowing into and out of a point on a closed circuit is zero.

We will analyze some circuits that consist of a single closed loop containing a resistor and either a capacitor or an inductor connected to a power source in series.



RC Circuit First we consider a circuit consisting of a simple loop containing a capacitor and a resistor in series with a voltage source $E(t)$, as illustrated by the following diagram. Suppose the capacitor holds an initial charge $Q_0 = 0$ and the switch is closed at time $t = 0$. What is the resulting current $I(t)$ in the closed loop for all times $t \geq 0$? Applying Kirchhoff's voltage law to the closed circuit gives $V_R + V_C = E(t)$, and using the voltage-current relationships for resistors and capacitors gives

$$RI + \frac{1}{C}\int_0^t I\,dt = E(t). \tag{1}$$

Consider the case where $E(t) = E_0$ is constant. Then Eq (1) becomes

$$RI + \frac{1}{C}\int_0^t I\,dt = E_0. \tag{2}$$

Notice that this equation is not a differential equation since it involves the integral of I. There are two ways to make this into a differential equation that we can solve. One option is to differentiate Eq (2) to reduce it to a differential equation

$$R\frac{dI}{dt} + \frac{I}{C} = 0. \tag{3}$$

The second way is to let $Q = \int_0^t I(s)\,ds$ be an antiderivative of $I(t)$. Then Eq (2) becomes the initial value problem

$$R\frac{dQ}{dt} + \frac{Q}{C} = E_0, \qquad Q(0) = 0. \tag{4}$$

We will first solve Eq (4). This is a constant coefficient linear first order differential equation in the unknown function Q. In standard form, it is

$$\frac{dQ}{dt} + \frac{Q}{RC} = \frac{E_0}{R} \tag{5}$$

so the integrating factor is $e^{t/RC}$. Multiplying Eq (5) by the integrating factor gives the equation

$$\frac{dQ}{dt}e^{t/RC} + \frac{Q}{RC}e^{t/RC} = \frac{E_0}{R}e^{t/RC} \tag{6}$$

which gives

$$\frac{d}{dt}\left(e^{t/RC}Q\right) = \frac{E_0}{R}e^{t/RC}. \tag{7}$$

Integrating gives

$$e^{t/RC}Q = E_0 C e^{t/RC} + K, \tag{8}$$

so that

$$Q = E_0 C + K e^{-t/RC}. \tag{9}$$

The initial condition $Q(0) = 0$ implies that $K = -E_0 C$ so the solution of the differential equation for Q is

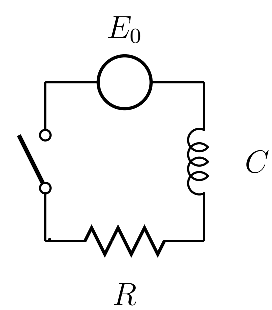
$$Q = E_0 C(1 - e^{-t/RC}). \tag{10}$$

Now differentiate this equation to get the current $I(t)$:

$$I(t) = \frac{dQ}{dt} = \frac{E_0}{R}e^{-t/RC}. \tag{11}$$

Note that the current decreases from its initial value of E_0/R to 0 as $t \to \infty$. As a second approach, solving Eq (3) with the initial condition $RI(0) = E_0$ obtained from Eq (2) by setting $t = 0$, we get exactly the same result.

RL Circuit Consider now the situation where an inductor and a resistor are present in a circuit, as in the following diagram, where the impressed voltage is a constant E_0. Kirchoff's voltage law then gives the governing equation



$$L\frac{dI}{dt} + RI = E_0, \qquad I(0) = 0. \tag{12}$$

The initial condition is obtained from the fact that the current in the system is zero at the instant when the switch is closed. This equation is almost the same as Eq. (4), where Q is replaced by I, R by L, and $1/C$ by R. Thus, the solution of Eq (12) is

$$I(t) = \frac{E_0}{R} - \frac{E_0}{R}e^{-Rt/L}. \tag{13}$$

Note that for this circuit, the current increases from its initial value of 0 to its limiting value $\lim_{t\to\infty} I(t) = E_0/R$.

Exercises

1. Consider an RC circuit with $R = 10$, $C = 2$, $E_0 = 0$. Suppose that the initial charge on the capacitor is $Q(0) = Q_0$. Find the charge $Q(t)$ for $t \geq 0$, and determine how long it takes for the charge to reach 10% of its initial value.

2. Suppose that the switch in an RC circuit with $E_0 = 0$ is closed at time $t = 0$ when the capacitor has a charge of Q_0. The current in the loop for time $t \geq 0$ is given by

$$I(t) = I_0 e^{-t/RC},$$

where $I_0 = I(0^+) = \lim t \to 0^+ I(t)$ is the "jump" in the current at time $t = 0$ when the switch is closed. What is the value of I_0? (*Hint:* $I(t) = Q'(t)$ for all $t > 0$, where $Q(t) = Q_0 e^{-t/RC}$.)

3. Consider an RC circuit with $R = 100$, $C = 0.1$, and $E_0 = 0$. Suppose that the initial charge on the capacitor is $Q(0) = 50$ and the switch is closed at time $t = 0$.

   (a) Find the charge for $t \geq 0$.

   (b) Compute the current $I(t) = Q'(t)$ for $t > 0$.

   (c) What is the "jump" in the current at $t = 0$.

Consider an RL circuit with a constant source voltage E_0 and initial current $I(0) = 0$. For the parameter values in each of Exercises 4 through 6,

(a) find the current $I(t)$ for $t \geq 0$;

(b) find $\lim_{t\to\infty} I(t)$;

(c) find the time at which $I(t) = \frac{9}{10} \lim_{t\to\infty} I(t)$.

4. $R = 100$; $L = 0.1$

5. $R = 10$; $L = 0.1$

6. $R = 10$; $L = 1$