Chapter 10
Fourier Series

10.1 Periodic Functions and Orthogonality Relations

The differential equation

$$y'' + \beta^2 y = F\cos\omega t$$

models a mass-spring system with natural frequency β with a pure cosine forcing function of frequency ω. If $\beta^2 \neq \omega^2$ a particular solution is easily found by undetermined coefficients (or by using Laplace transforms) to be

$$y_p = \frac{F}{\beta^2 - \omega^2}\cos\omega t.$$

If the forcing function is a linear combination of simple cosine functions, so that the differential equation is

$$y'' + \beta^2 y = \sum_{n=1}^{N} F_n \cos\omega_n t$$

where $\beta^2 \neq \omega_n^2$ for any n, then, by linearity, a particular solution is obtained as a sum

$$y_p(t) = \sum_{n=1}^{N} \frac{F_n}{\beta^2 - \omega_n^2}\cos\omega_n t.$$

This simple procedure can be extended to any function that can be represented as a sum of cosine (and sine) functions, even if that summation is not a finite sum. It turns out that the functions that can be represented as sums in this form are very general, and include most of the periodic functions that are usually encountered in applications.

Periodic Functions

A function f is said to be **periodic with period** $p > 0$ if

$$f(t+p) = f(t)$$

for all t in the domain of f. This means that the graph of f repeats in successive intervals of length p, as can be seen in the graph in Figure 10.1.

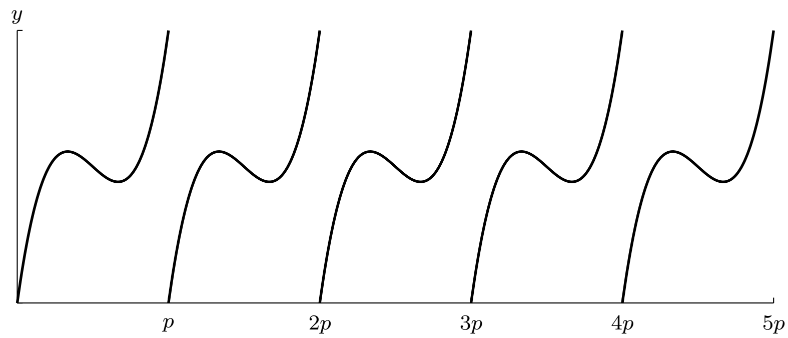


Fig. 10.1 An example of a periodic function with period p. Notice how the graph repeats on each interval of length p.

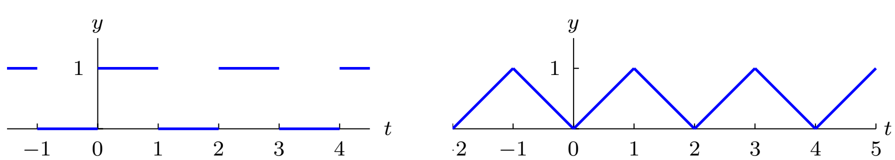
The functions $\sin t$ and $\cos t$ are periodic with period 2π, while $\tan t$ is periodic with period π since

$$\tan(t+\pi) = \frac{\sin(t+\pi)}{\cos(t+\pi)} = \frac{-\sin t}{-\cos t} = \tan t.$$

The constant function $f(t) = c$ is periodic with period p where p is *any* positive number since

$$f(t+p) = c = f(t).$$

Other examples of periodic functions are the square wave and triangular wave whose graphs are shown in Figure 10.2. Both are periodic with period 2.



(a) Square wave $\mathrm{sw}(t)$ (b) Triangular wave $\mathrm{tw}(t)$

Fig. 10.2

Since a periodic function of period p repeats over any interval of length p, it is possible to define a periodic function by giving the formula for f on an interval of length p, and repeating this in subsequent intervals of length p. For example, the square wave $\mathrm{sw}(t)$ and triangular wave $\mathrm{tw}(t)$ from Figure 10.2 are described by

$$\mathrm{sw}(t) = \begin{cases} 0 & \text{if } -1 \le t < 0 \\ 1 & \text{if } 0 \le t < 1 \end{cases}; \qquad \mathrm{sw}(t+2) = \mathrm{sw}(t).$$

$$\mathrm{tw}(t) = \begin{cases} -t & \text{if } -1 \le t < 0 \\ t & \text{if } 0 \le t < 1 \end{cases}; \qquad \mathrm{tw}(t+2) = \mathrm{tw}(t).$$

There is not a unique period for a periodic function. Note that if p is a period of $f(t)$, then $2p$ is also a period because

$$f(t+2p) = f((t+p)+p) = f(t+p) = f(t)$$

for all t. In fact, a similar argument shows that np is also a period for any positive integer n. Thus $2n\pi$ is a period for $\sin t$ and $\cos t$ for all positive integers n.

If $P > 0$ is a period of f and there is no smaller period then we say P is the **fundamental period** of f although we will usually just say **the period**. Not all periodic functions have a smallest period. The constant function $f(t) = c$ is an example of such a function since any positive p is a period. The fundamental period of the sine and cosine functions is 2π, while the fundamental period of the square wave and triangular wave from Figure 10.2 is 2.

Periodic functions under scaling

If $f(t)$ is periodic of period p and a is any positive number let $g(t) = f(at)$. Then for all t

$$g\big(t + \frac{p}{a}\big) = f\big(a\big(t + \frac{p}{a}\big)\big) = f(at+p) = f(at) = g(t).$$

Thus

$$\boxed{\text{If } f(t) \text{ is periodic with period } p, \text{ then } f(at) \text{ is periodic with period } \frac{p}{a}.}$$

It is also true that if P is the *fundamental period* for the periodic function $f(t)$, then the *fundamental period* of $g(t) = f(at)$ is P/a. To see this it is only necessary to verify that any period r of $g(t)$ is at least as large as P/a, which is already a period as observed above. But if r is a period for $g(t)$, then ra is a period for $g(t/a) = f(t)$ so that $ra \ge P$, or $r \ge P/a$.

Applying these observations to the functions $\sin t$ and $\cos t$ with fundamental period 2π gives the following facts.

Theorem 1. *For any $a > 0$ the functions $\cos at$ and $\sin at$ are periodic with period $2\pi/a$. In particular, if $L > 0$ then the functions*

$$\cos \frac{n\pi}{L}t \quad \textit{and} \quad \sin \frac{n\pi}{L}t, \qquad n = 1, 2, 3, \ldots .$$

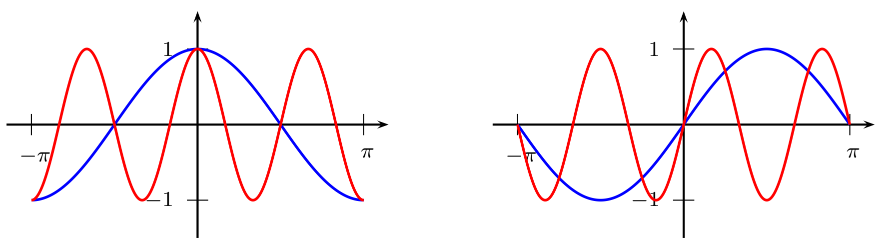
are periodic with fundamental period $P = 2L/n$.

Note that since the fundamental period of the functions $\cos \frac{n\pi}{L}t$ and $\sin \frac{n\pi}{L}t$ is $P = 2L/n$, it follows that $2L = nP$ is also a period for each of these functions. Thus, a sum

$$\sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L}t + b_n \sin \frac{n\pi}{L}t\right)$$

will be periodic of period $2L$.

Notice that if n is a positive integer, then $\cos nt$ and $\sin nt$ are periodic with period $2\pi/n$. Thus, each period of $\cos t$ or $\sin t$ contains n periods of $\cos nt$ and $\sin nt$. This means that the functions $\cos nt$ and $\sin nt$ oscillate more rapidly as n increases, as can be seen in Figure 10.3 for $n = 3$.



(a) The graphs of $\cos t$ and $\cos 3t$.

(b) The graphs of $\sin t$ and $\sin 3t$.

Fig. 10.3

Example 2. Find the fundamental period of each of the following periodic functions.

1. $\cos 2t$
2. $\sin \frac{3}{2}(t - \pi)$
3. $1 + \cos t + \cos 2t$
4. $\sin 2\pi t + \sin 3\pi t$

▶ Solution. 1. $P = 2\pi/2 = \pi$.

2. $\sin \frac{3}{2}(t-\pi) = \sin(\frac{3}{2}t - \frac{3}{2}\pi) = \sin \frac{3}{2}t \cos \frac{3}{2}\pi - \cos \frac{3}{2}t \sin \frac{3}{2}\pi = \cos \frac{3}{2}t$. Thus, $P = 2\pi/(3/2) = 4\pi/3$.

3. The constant function 1 is periodic with any period p, the fundamental period of $\cos t$ is 2π and all the periods are of the form $2n\pi$ for a positive integer n, and the fundamental period of $\cos 2t$ is π with $m\pi$ being all the possible periods. Thus, the smallest number that works as a period for all the functions is 2π and this is also the smallest period for the sum. Hence $P = 2\pi$.
4. The fundamental period of $\sin 2\pi t$ is $2\pi/2\pi = 1$ so that all of the periods have the form n for n a positive integer, and the fundamental period of $\sin 3\pi t$ is $2\pi/3\pi = 2/3$ so that all of the periods have the form $2m/3$ for m a positive integer. Thus, the smallest number that works as a period for both functions is 2 and thus $P = 2$.

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Orthogonality Relations for Sine and Cosine

The family of linearly independent functions

$$\left\{1,\, \cos\frac{\pi}{L}t,\, \cos\frac{2\pi}{L}t,\, \cos\frac{3\pi}{L}t,\, \ldots,\, \sin\frac{\pi}{L}t,\, \sin\frac{2\pi}{L}t,\, \sin\frac{3\pi}{L}t,\, \ldots\right\}$$

form what is called a mutually orthogonal set of functions on the interval $[-L, L]$, analogous to a mutually perpendicular set of vectors. Two functions f and g defined on an interval $a \le t \le b$ are said to be **orthogonal** on the interval $[a, b]$ if

$$\int_a^b f(t)g(t)\,dt = 0.$$

A family of functions is **mutually orthogonal** on the interval $[a, b]$ if any two distinct functions are orthogonal. The mutual orthogonality of the family of cosine and sine functions on the interval $[-L, L]$ is a consequence of the following identities.

Proposition 3 (Orthogonality Relations). *Let m and n be positive integers, and let $L > 0$. Then*

$$\int_{-L}^{L} \cos\frac{n\pi}{L}t\,dt = \int_{-L}^{L} \sin\frac{n\pi}{L}t\,dt = 0 \tag{1}$$

$$\int_{-L}^{L} \cos\frac{n\pi}{L}t \sin\frac{m\pi}{L}t\,dt = 0 \tag{2}$$

$$\int_{-L}^{L} \cos\frac{n\pi}{L}t \cos\frac{m\pi}{L}t\,dt = \begin{cases} L, & \text{if } n = m, \\ 0 & \text{if } n \ne m. \end{cases} \tag{3}$$

$$\int_{-L}^{L} \sin\frac{n\pi}{L}t \sin\frac{m\pi}{L}t\,dt = \begin{cases} L, & \text{if } n = m, \\ 0 & \text{if } n \ne m. \end{cases} \tag{4}$$

Proof. For (1):

$$\int_{-L}^{L} \cos \frac{n\pi}{L} t\,dt = \frac{L}{n\pi} \sin \frac{n\pi}{L} t \Big|_{-L}^{L} = \frac{L}{n\pi}(\sin n\pi - \sin(-n\pi)) = 0.$$

For (3) with $n \neq m$, use the identity

$$\cos A \cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B)),$$

to get

$$\int_{-L}^{L} \cos \frac{n\pi}{L} t \cos \frac{m\pi}{L} t\,dt = \int_{-L}^{L} \frac{1}{2}\left(\cos \frac{(n+m)\pi}{L} t + \cos \frac{(n-m)\pi}{L} t\right) dt$$
$$= \left(\frac{L}{2(n+m)\pi} \sin \frac{(n+m)\pi}{L} t + \frac{L}{2(n-m)\pi} \sin \frac{(n-m)\pi}{L} t\right)\Big|_{-L}^{L} = 0.$$

For (3) with $n = m$, use the identity $\cos^2 A = (1 + \cos 2A)/2$ to get

$$\int_{-L}^{L} \cos \frac{n\pi}{L} t \cos \frac{m\pi}{L} t\,dt = \int_{-L}^{L} \left(\cos \frac{n\pi}{L} t\right)^2 dt =$$
$$= \int_{-L}^{L} \frac{1}{2}\left(1 + \cos \frac{2n\pi}{L} t\right) dt = \frac{1}{2}\left(t + \frac{L}{2n\pi} \sin \frac{2n\pi}{L} t\right)\Big|_{-L}^{L} = L.$$

The proof of (4) is similar, making use of the identities $\sin^2 A = (1-\cos 2A)/2$ in case $n = m$ and

$$\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$$

in case $n \neq m$. The proof of (2) is left as an exercise.

Even and Odd Functions

A function f defined on a symmetric interval $[-L, L]$ is **even** if $f(-t) = f(t)$ for $-L \leq t \leq L$, and f is **odd** if $f(-t) = -f(t)$ for $-L \leq t \leq L$.

Example 4. Determine whether each of the following functions is even, odd, or neither.

1. $f(t) = 3t^2 + \cos 5t$
2. $g(t) = 3t - t^2 \sin 2t$
3. $h(t) = t^2 + t + 1$

► Solution. 1. Since $f(-t) = 3(-t)^2 + \cos 5(-t) = 3t^2 + \cos 5t = f(t)$ for all t, it follows that f is an even function.

2. Since $g(-t) = 3(-t) - (-t)^2 \sin 3(-t) = -3t + t^2 \sin 3t = -g(t)$ for all t, it follows that g is an odd function.
3. Since $h(-t) = (-t)^2 + (-t) + 1 = t^2 - t + 1 = t^2 + t + 1 = h(t) \iff -t = t \iff t = 0$, we conclude that h is not even. Similarly, $h(-t) = -h(t) \iff t^2 - t + 1 = -t^2 - t - 1 \iff t^2 + 1 = (-t^2 + 1) \iff t^2 + 1 = 0$, which is not true for any t. Thus h is not odd, and hence it is neither even or odd.

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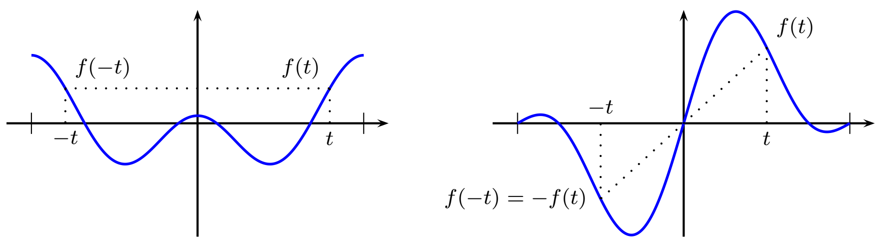
The graph of an even function is symmetric with respect to the y-axis, while the graph of an odd function is symmetric with respect to the origin, as illustrated in Figure 10.4.



(a) The graph of an even function.

(b) The graph of an odd function.

Fig. 10.4

Here is a list of basic properties of even and odd functions that are useful in applications to Fourier series. All of them follow easily from the definitions, and the verifications will be left to the exercises.

Proposition 5. *Suppose that f and g are functions defined on the interval $-L \leq t \leq L$.*

1. *If both f and g are even then $f + g$ and fg are even.*
2. *If both f and g are odd, then $f + g$ is odd and fg is even.*
3. *If f is even and g is odd, then fg is odd.*
4. *If f is even, then*
$$\int_{-L}^{L} f(t)\,dt = 2\int_{0}^{L} f(t)\,dt.$$
5. *If f is odd, then*
$$\int_{-L}^{L} f(t)\,dt = 0.$$

Since the integral of f computes the signed area under the graph of f, the integral equations can be seen from the graphs of even and odd functions in Figure 10.4.

Exercises

1–9. Graph each of the following periodic functions. Graph at least 3 periods.

1. $f(t) = \begin{cases} 3 & \text{if } 0 < t < 3 \\ -3 & \text{if } -3 < t < 0 \end{cases}; \qquad f(t+6) = f(t).$

2. $f(t) = \begin{cases} -3 & \text{if } -2 \le t < -1 \\ 0 & \text{if } -1 \le t \le 1 \\ 3 & \text{if } 1 < t < 2 \end{cases}; \qquad f(t+4) = f(t).$

3. $f(t) = t, \quad 0 < t \le 2; \quad f(t+2) = f(t).$

4. $f(t) = t, \quad -1 < t \le 1; \quad f(t+2) = f(t).$

5. $f(t) = \sin t, \quad 0 < t \le \pi; \quad f(t+\pi) = f(t).$

6. $f(t) = \begin{cases} 0 & \text{if } -\pi \le t < 0 \\ \sin t & \text{if } 0 < t \le \pi \end{cases}; \quad f(t+2\pi) = f(t).$

7. $f(t) = \begin{cases} -t & \text{if } -1 \le t < 0 \\ 1 & \text{if } 0 \le t < 1 \end{cases}; \qquad f(t+2) = f(t).$

8. $f(t) = t^2, \quad -1 < t \le 1; \quad f(t+2) = f(t).$

9. $f(t) = t^2, \quad 0 < t \le 2; \quad f(t+2) = f(t).$

10–17. Determine if the given function is periodic. If it is periodic find the fundamental period.

10. 1

11. $\sin 2t$

12. $1 + \cos 3\pi t$

13. $\cos 2t + \sin 3t$

14. $t + \sin 2t$

15. $\sin^2 t$

16. $\cos t + \cos \pi t$

17. $\sin t + \sin 2t + \sin 3t$

18–26. Determine if the given function is even, odd, or neither.

18. $f(t) = |t|$

19. $f(t) = t\,|t|$

20. $f(t) = \sin^2 t$

21. $f(t) = \cos^2 t$

22. $f(t) = \sin t \sin 3t$

23. $f(t) = t^2 + \sin t$

24. $f(t) = t + |t|$

25. $f(t) = \ln|\cos t|$

26. $f(t) = 5t + t^2 \sin 3t$

27. Verify the orthogonality property (2) from Proposition 3:

$$\int_{-L}^{L} \cos\frac{n\pi}{L}t \sin\frac{m\pi}{L}t\,dt = 0$$

28. Use the properties of even and odd functions (Proposition 10.4) to evaluate the following integrals.

(a) $\displaystyle\int_{-1}^{1} t\,dt$

(b) $\displaystyle\int_{-1}^{1} t^4\,dt$

(c) $\displaystyle\int_{-\pi}^{\pi} t\sin t\,dt$

(d) $\displaystyle\int_{-\pi}^{\pi} t\cos t\,dt$

(e) $\displaystyle\int_{-\pi}^{\pi} \cos\frac{n\pi}{L}t \sin\frac{m\pi}{L}t\,dt$

(f) $\displaystyle\int_{-\pi}^{\pi} t^2\sin t\,dt$

10.2 Fourier Series

We start by considering the possibility of representing a function f as a sum of a series of the form

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi t}{L} + b_n\sin\frac{n\pi t}{L}\right) \qquad (1)$$

where the coefficients a_0, a_1, ..., b_1, b_2, ..., are to be determined. Since the individual terms in the series (1) are periodic with periods $2L$, $2L/2$, $2L/3$, ..., the function $f(t)$ determined by the sum of the series, where it converges, must be periodic with period $2L$. This means that only periodic functions of period $2L$ can be represented by a series of the form (1). Our first problem is to find the coefficients a_n and b_n in the series (1). The first term of the series is written $a_0/2$, rather than simply as a_0, to make the formula to be derived below the same for all a_n, rather than a special case for a_0.

The coefficients a_n and b_n can be found from the orthogonality relations of the family of functions $\cos(n\pi t/L)$ and $\sin(n\pi t/L)$ on the interval $[-L, L]$ given in Proposition 3 of Sect. 10.1. To compute the coefficient a_n for $n = 1, 2, 3, \ldots$, multiply both sides of the series (1) by $\cos(m\pi t/L)$, with m a positive integer and then integrate from $-L$ to L. For the moment we will assume that the integrals exist and that it is justified to integrate term by term. Then using (1), (2), and (3) from Sect. 10.1, we get

$$\begin{aligned}\int_{-L}^{L} f(t)\cos\frac{m\pi}{L}t\,dt &= \frac{a_0}{2}\underbrace{\int_{-L}^{L}\cos\frac{m\pi}{L}t\,dt}_{=0}\\ &+\sum_{n=1}^{\infty}\Big[a_n\underbrace{\int_{-L}^{L}\cos\frac{n\pi}{L}t\cos\frac{m\pi}{L}t\,dt}_{=\begin{cases}0 & \text{if } n\neq m\\ L & \text{if } n=m\end{cases}}+b_n\underbrace{\int_{-L}^{L}\sin\frac{n\pi}{L}t\cos\frac{m\pi}{L}t\,dt}_{=0}\Big]=a_mL.\end{aligned}$$

Thus,

$$a_m = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{m\pi}{L}t\,dt, \qquad m = 1, 2, 3, \ldots$$

or, replacing the index m by n,

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi}{L}t\,dt, \qquad n = 1, 2, 3, \ldots \tag{2}$$

To compute a_0, integrate both sides of (1) from $-L$ to L to get

$$\begin{aligned}\int_{-L}^{L} f(t)\,dt &= \frac{a_0}{2}\underbrace{\int_{-L}^{L}dt}_{=2L}+\sum_{n=1}^{\infty}\Big[a_n\underbrace{\int_{-L}^{L}\cos\frac{n\pi}{L}t\,dt}_{=0}+b_n\underbrace{\int_{-L}^{L}\sin\frac{n\pi}{L}t\,dt}_{=0}\Big]\\ &= a_0L.\end{aligned}$$

Thus,

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt. \tag{3}$$

Hence, a_0 is two times the average value of the function $f(t)$ over the interval $-L \le t \le L$. Observe that the value of a_0 is obtained from (2) by setting $n = 0$. Of course, if the constant a_0 in (1) were not divided by 2, we would need a separate formula for a_0. It is for this reason that the constant term in (1) is labeled $a_0/2$. Thus, for all $n \ge 0$, the coefficients a_n are given by a single formula

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi}{L}t\,dt, \qquad n = 0,\,1,\,2,\,\ldots \tag{4}$$

To compute b_n for $n = 1, 2, 3, \ldots$, multiply both sides of the series (1) by $\sin(m\pi t/L)$, with m a positive integer and then integrate from $-L$ to L. Then using (1), (2), and (4) from Sect. 10.1, we get

$$\begin{aligned}\int_{-L}^{L} f(t)\sin\frac{m\pi}{L}t\,dt &= \frac{a_0}{2}\underbrace{\int_{-L}^{L}\sin\frac{m\pi}{L}t\,dt}_{=0}\\ &+\sum_{n=1}^{\infty}\Bigg[a_n\underbrace{\int_{-L}^{L}\cos\frac{n\pi}{L}t\sin\frac{m\pi}{L}t\,dt}_{=0}+b_n\underbrace{\int_{-L}^{L}\sin\frac{n\pi}{L}t\sin\frac{m\pi}{L}t\,dt}_{=\begin{cases}0 & \text{if } n\ne m\\ L & \text{if } n=m\end{cases}}\Bigg]=b_mL.\end{aligned}$$

Thus, replacing the index m by n, we find that

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi}{L}t\,dt, \qquad n = 1,\,2,\,3,\,\ldots \tag{5}$$

We have arrived at what are known as the **Euler Formulas** for a function $f(t)$ that is the sum of a trigonometric series as in (1):

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt \tag{6}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi}{L}t\,dt, \qquad n = 1,\,2,\,3,\,\ldots \tag{7}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi}{L}t\,dt, \qquad n = 1,\,2,\,3,\,\ldots \tag{8}$$

The numbers a_n and b_n are known as the **Fourier coefficients** of the function f. Note that while we started with a periodic function of period $2L$, the formulas for a_n and b_n only use the values of $f(t)$ on the interval $[-L,\,L]$.

We can then reverse the process, and start with any function $f(t)$ defined on the symmetric interval $[-L, L]$ and use the Euler formulas to determine a trigonometric series. We will write

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi t}{L} + b_n \sin \frac{n\pi t}{L} \right), \tag{9}$$

where the a_n, b_n are defined by (6), (7), and (8), to indicate that the right hand side of (9) is the **Fourier series** of the function $f(t)$ defined on $[-L, L]$. Note that the symbol $\sim$ indicates that the trigonometric series on the right of (9) is associated with the function $f(t)$; it does not imply that the series converges to $f(t)$ for any value of t. In fact, there are functions whose Fourier series do not converge to the function. Of course, we will be interested in the conditions under which the Fourier series of $f(t)$ converges to $f(t)$, in which case $\sim$ can be replaced by $=$; but for now we associate a specific series using Equations (6), (7), and (8) with $f(t)$ and call it the Fourier series. The mild conditions under which the Fourier series of $f(t)$ converges to $f(t)$ will be considered in the next section.

Remark 1. If an integrable function $f(t)$ is periodic with period p, then the integral of $f(t)$ over any interval of length p is the same; that is

$$\int_c^{c+p} f(t)\,dt = \int_0^p f(t)\,dt \tag{10}$$

for any choice of c. To see this, first observe that for any α and β, if we use the change of variables $t = x - p$, then

$$\int_\alpha^\beta f(t)\,dt = \int_{\alpha+p}^{\beta+p} f(x-p)\,dx = \int_{\alpha+p}^{\beta+p} f(x)\,dx = \int_{\alpha+p}^{\beta+p} f(t)\,dt.$$

Letting $\alpha = c$ and $\beta = 0$ gives

$$\int_c^0 f(t)\,dt = \int_{c+p}^p f(t)\,dt$$

so that

$$\begin{aligned}\int_c^{c+p} f(t)\,dt &= \int_c^0 f(t)\,dt + \int_0^{c+p} f(t)\,dt \\ &= \int_{c+p}^p f(t)\,dt + \int_0^{c+p} f(t)\,dt = \int_0^p f(t)\,dt,\end{aligned}$$

which is (10). This formula means that when computing the Fourier coefficients, the integrals can be computed over any convenient interval of length $2L$. For example,

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi}{L}t\,dt = \frac{1}{L}\int_{0}^{2L} f(t)\cos\frac{n\pi}{L}t\,dt.$$

We now consider some examples of the calculation of Fourier series.

Example 2. Compute the Fourier series of the odd square wave function of period $2L$ and amplitude 1 given by

$$f(t) = \begin{cases} -1 & -L \le t < 0, \\ 1 & 0 \le t < L, \end{cases}; \qquad f(t+2L) = f(t).$$

See Figure 10.5 for the graph of $f(t)$.

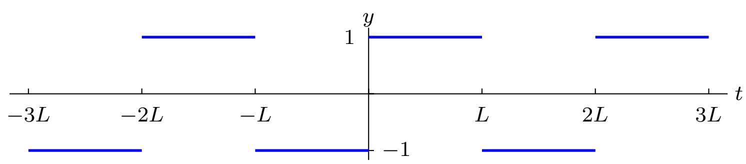


Fig. 10.5 The odd square wave of period $2L$

▶ Solution. Use the Euler formulas for a_n (Equations (6) and (7)) to conclude

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi}{L}t\,dt = 0$$

for all $n \ge 0$. This is because the function $f(t)\cos\frac{n\pi}{L}t$ is the product of an odd and even function, and hence is odd, which implies by Proposition 5 (Part 5) of Section 10.1 that the integral is 0. It remains to compute the coefficients b_n from (8).

$$\begin{aligned}
b_n &= \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi}{L}t\,dt = \frac{1}{L}\int_{-L}^{0} f(t)\sin\frac{n\pi}{L}t\,dt + \frac{1}{L}\int_{0}^{L} f(t)\sin\frac{n\pi}{L}t\,dt \\
&= \frac{1}{L}\int_{-L}^{0} (-1)\sin\frac{n\pi}{L}t\,dt + \frac{1}{L}\int_{0}^{L} (+1)\sin\frac{n\pi}{L}t\,dt \\
&= \frac{1}{L}\left\{\left[\frac{L}{n\pi}\cos\frac{n\pi}{L}t\right]_{-L}^{0} - \left[\frac{L}{n\pi}\cos\frac{n\pi}{L}t\right]_{0}^{L}\right\} \\
&= \frac{1}{n\pi}[(1-\cos(-n\pi)) + (1-\cos(n\pi))] \\
&= \frac{2}{n\pi}(1-\cos n\pi) = \frac{2}{n\pi}(1-(-1)^n).
\end{aligned}$$

Therefore,

$$b_n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{4}{n\pi} & \text{if } n \text{ is odd,} \end{cases}$$

and the Fourier series is

$$f(t) \sim \frac{4}{\pi}\left(\sin\frac{\pi}{L}t + \frac{1}{3}\sin\frac{3\pi}{L}t + \frac{1}{5}\sin\frac{5\pi}{L}t + \frac{1}{7}\sin\frac{7\pi}{L}t + \cdots\right). \tag{11}$$

◀

Example 3. Compute the Fourier series of the even square wave function of period $2L$ and amplitude 1 given by

$$f(t) = \begin{cases} -1 & -L \le t < -L/2, \\ 1 & -L/2 \le t < L/2, ; \\ -1 & L/2 \le t < L, \end{cases} \qquad f(t+2L) = f(t).$$

See Figure 10.6 for the graph of $f(t)$.

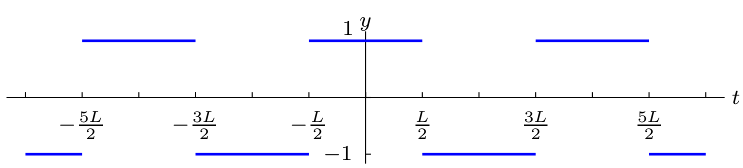


Fig. 10.6 The even square wave of period $2L$

► Solution. Use the Euler formulas for b_n (Equation (8)) to conclude

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi}{L}t\,dt = 0,$$

for all $n \ge 1$. As in the previous example, this is because the function $f(t)\sin\frac{n\pi}{L}t$ is the product of an even and odd function, and hence is odd, which implies by Proposition 5 (Part 5) of Section 10.1 that the integral is 0. It remains to compute the coefficients a_n from (6) and (7).

For $n = 0$, a_0 is twice the average of $f(t)$ over the period $[-L, L]$, which is easily seen to be 0 from the graph of $f(t)$. For $n \ge 1$,

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi}{L}t\,dt \\ &= \frac{1}{L}\int_{-L}^{-L/2} f(t)\cos\frac{n\pi}{L}t\,dt + \frac{1}{L}\int_{-L/2}^{L/2} f(t)\cos\frac{n\pi}{L}t\,dt + \frac{1}{L}\int_{L/2}^{L} f(t)\cos\frac{n\pi}{L}t\,dt \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{L}\int_{-L}^{-L/2}(-1)\cos\frac{n\pi}{L}t\,dt + \frac{1}{L}\int_{-L/2}^{L/2}(+1)\cos\frac{n\pi}{L}t\,dt + \frac{1}{L}\int_{L/2}^{L}(-1)\cos\frac{n\pi}{L}t\,dt \\
&= \frac{1}{L}\left\{-\left[\frac{L}{n\pi}\sin\frac{n\pi}{L}t\right]_{-L}^{-L/2} + \left[\frac{L}{n\pi}\sin\frac{n\pi}{L}t\right]_{-L/2}^{L/2} - \left[\frac{L}{n\pi}\sin\frac{n\pi}{L}t\right]_{L/2}^{L}\right\} \\
&= \frac{1}{n\pi}\Big[-\sin\big(-\frac{n\pi}{2}\big) + \sin(-n\pi) + \sin\big(\frac{n\pi}{2}\big) - \sin\big(-\frac{n\pi}{2}\big) - \sin n\pi + \sin\big(\frac{n\pi}{2}\big)\Big] \\
&= \frac{4}{n\pi}\sin\frac{n\pi}{2}.
\end{aligned}
$$

Therefore,

$$
a_n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{4}{n\pi} & \text{if } n = 4m+1 \\ -\frac{4}{n\pi} & \text{if } n = 4m+3 \end{cases}
$$

and the Fourier series is

$$
\begin{aligned}
f(t) &\sim \frac{4}{\pi}\left(\cos\frac{\pi}{L}t - \frac{1}{3}\cos\frac{3\pi}{L}t + \frac{1}{5}\cos\frac{5\pi}{L}t - \frac{1}{7}\cos\frac{7\pi}{L}t + \cdots\right) \\
&= \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cos\frac{(2k+1)\pi}{L}t.
\end{aligned}
$$

◀

Example 4. Compute the Fourier series of the even triangular wave function of period 2π given by

$$
f(t) = \begin{cases} -t & -\pi \le t < 0, \\ t & 0 \le t < \pi, \end{cases}; \qquad f(t+2\pi) = f(t).
$$

See Figure 10.7 for the graph of $f(t)$.

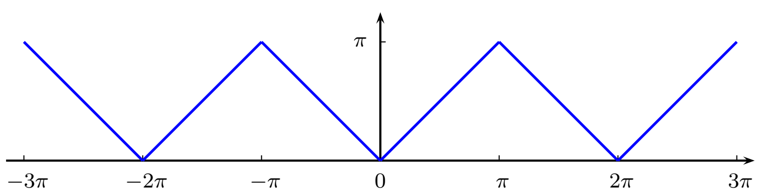


Fig. 10.7 The even triangular wave of period 2π.

▶ Solution. The period is $2\pi = 2L$ so $L = \pi$. Again, since the function $f(t)$ is even, the coefficients $b_n = 0$. It remains to compute the coefficients a_n from the Euler formulas (6) and (7).

For $n = 0$, using the fact that $f(t)$ is even,

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{2}{\pi}\int_0^{\pi} f(t)\,dt \\ &= \frac{2}{\pi}\int_0^{\pi} t\,dt = \frac{2}{\pi}\,\frac{t^2}{2}\bigg|_0^{\pi} = \pi. \end{aligned}$$

For $n \geq 1$, using the fact that $f(t)$ is even, and taking advantage of the integration by parts formula

$$\int x\cos x\,dx = x\sin x + \cos x + C,$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt = \frac{2}{\pi}\int_0^{\pi} f(t)\cos nt\,dt \\ &= \frac{2}{\pi}\int_0^{\pi} t\cos nt\,dt \qquad \left(\text{let } x = nt \text{ so } t = \frac{x}{n} \text{ and } dt = \frac{dx}{n}\right) \\ &= \frac{2}{\pi}\int_0^{n\pi} \frac{x}{n}\cos x\,\frac{dx}{n} = \frac{2}{n^2\pi}\left[x\sin x + \cos x\right]_{x=0}^{x=n\pi} \\ &= \frac{2}{n^2\pi}[\cos n\pi - 1] = \frac{2}{n^2\pi}[(-1)^n - 1] \end{aligned}$$

Therefore,

$$a_n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ -\frac{4}{n^2\pi} & \text{if } n \text{ is odd} \end{cases}$$

and the Fourier series is

$$\begin{aligned} f(t) &\sim \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos t}{1^2} + \frac{\cos 3t}{3^2} + \frac{\cos 5t}{5^2} + \frac{\cos 7t}{7^2} + \cdots\right) \\ &= \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\cos(2k+1)t}{(2k+1)^2}. \end{aligned}$$

◀

Example 5. Compute the Fourier series of the sawtooth wave function of period $2L$ given by

$$f(t) = t \quad \text{for } -L \leq t < L; \qquad f(t+2L) = f(t).$$

See Figure 10.8 for the graph of $f(t)$.

► Solution. As in Example 2, the function $f(t)$ is odd, so the cosine terms a_n are all 0. Now compute the coefficients b_n from (8). Using the integration by parts formula

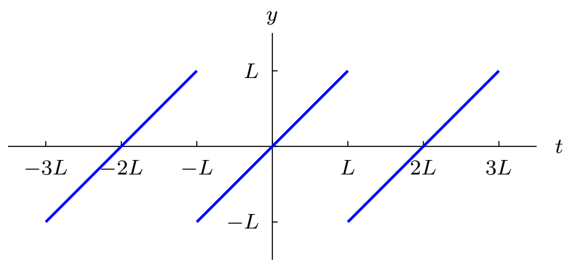


Fig. 10.8 The sawtooth wave of period $2L$

$$\int x \sin x \, dx = \sin x - x \cos x + C,$$

$$\begin{aligned}
b_n &= \frac{1}{L} \int_{-L}^{L} f(t) \sin \frac{n\pi}{L} t \, dt \\
&= \frac{2}{L} \int_0^L t \sin \frac{n\pi}{L} t \, dt \qquad \left(\text{let } x = \frac{n\pi}{L} t \text{ so } t = \frac{L}{n\pi} x \text{ and } dt = \frac{L}{n\pi} dx \right) \\
&= \frac{2}{L} \int_0^{n\pi} \frac{L}{n\pi} x \sin x \, \frac{L\, dx}{n\pi} = \frac{2L}{n^2\pi^2} \int_0^{n\pi} x \sin x \, dx \\
&= \frac{2L}{n^2\pi^2} \left[\sin x - x \cos x \right]_{x=0}^{x=n\pi} \\
&= -\frac{2L}{n^2\pi^2} (n\pi \cos n\pi) = -\frac{2L}{n\pi} (-1)^n.
\end{aligned}$$

Therefore, the Fourier series is

$$\begin{aligned}
f(t) &\sim \frac{2L}{\pi} \left(\sin \frac{\pi}{L} t - \frac{1}{2} \sin \frac{2\pi}{L} t + \frac{1}{3} \sin \frac{3\pi}{L} t - \frac{1}{4} \sin \frac{4\pi}{L} t + \cdots \right) \\
&= \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi}{L} t.
\end{aligned}$$

◀

All of the examples so far have been of functions that are either even or odd. If a function $f(t)$ is even, the resulting Fourier series will only have cosine terms, as in the case of Examples 3 and 4, while if $f(t)$ is odd, the resulting Fourier series will only have sine terms, as in Examples 2 and 5. Here are some examples where both sine an cosine terms appear.

Example 6. Compute the Fourier series of the function of period 4 given by

$$f(t) = \begin{cases} 0 & -2 \le t < 0, \\ t & 0 \le t < 2, \end{cases}; \qquad f(t+4) = f(t).$$

See Figure 10.9 for the graph of $f(t)$.

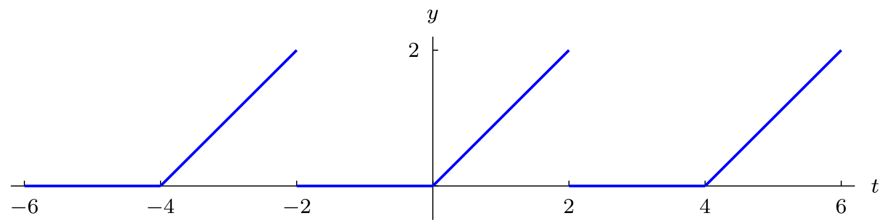


Fig. 10.9 A half sawtooth wave of period 4

► Solution. This function is neither even nor odd, so we expect both sine and cosine terms to be present. The period is $4 = 2L$ so $L = 2$. Because $f(t) = 0$ on the interval $(-2, 0)$, each of the integrals in the Euler formulas, which should be an integral from $t = -2$ to $t = 2$, can be replaced with an integral from $t = 0$ to $t = 2$. Thus, the Euler formulas give

$$\begin{aligned} a_0 &= \frac{1}{2}\int_0^2 t\,dt = \frac{1}{2}\left[\frac{t^2}{2}\right]_0^2 = 1; \\ a_n &= \frac{1}{2}\int_0^2 t\cos\frac{n\pi}{2}t\,dt \qquad \left(\text{let } x = \frac{n\pi}{2}t \text{ so } t = \frac{2x}{n\pi} \text{ and } dt = \frac{2dx}{n\pi}\right) \\ &= \frac{1}{2}\int_0^{n\pi} \frac{2x}{n\pi}\cos x\,\frac{2dx}{n\pi} = \frac{2}{n^2\pi^2}\int_0^{n\pi} x\cos x\,dx \\ &= \frac{2}{n^2\pi^2}\left[\cos x + x\sin x\right]_{x=0}^{x=n\pi} \\ &= \frac{2}{n^2\pi^2}(\cos n\pi - 1) = \frac{2}{n^2\pi^2}((-1)^n - 1). \end{aligned}$$

Therefore,

$$a_n = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \text{ is even and } n \ge 2, \\ -\frac{4}{n^2\pi^2} & \text{if } n \text{ is odd.} \end{cases}$$

Now compute b_n:

$$b_n = \frac{1}{2}\int_0^2 t\sin\frac{n\pi}{2}t\,dt \qquad \left(\text{let } x = \frac{n\pi}{2}t \text{ so } t = \frac{2x}{n\pi} \text{ and } dt = \frac{2dx}{n\pi}\right)$$

$$= \frac{1}{2}\int_0^{n\pi} \frac{2x}{n\pi}\sin x\,\frac{2dx}{n\pi} = \frac{2}{n^2\pi^2}\int_0^{n\pi} x\sin x\,dx$$
$$= \frac{2}{n^2\pi^2}\left[\sin x - x\cos x\right]_{x=0}^{x=n\pi}$$
$$= \frac{2}{n^2\pi^2}(-n\pi\cos n\pi) = -\frac{2}{n\pi}(-1)^n.$$

Thus,

$$b_n = \frac{2(-1)^{n+1}}{n\pi} \qquad \text{for all } n \geq 1.$$

Therefore, the Fourier series is

$$f(t) \sim \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos\frac{(2k+1)\pi}{2}t + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi}{2}t.$$

◀

Example 7. Compute the Fourier series of the square pulse wave function of period 2π given by

$$f(t) = \begin{cases} 1 & 0 \leq t < h, \\ 0 & h \leq t < 2\pi, \end{cases}; \qquad f(t+2\pi) = f(t).$$

See Figure 10.10 for the graph of $f(t)$.

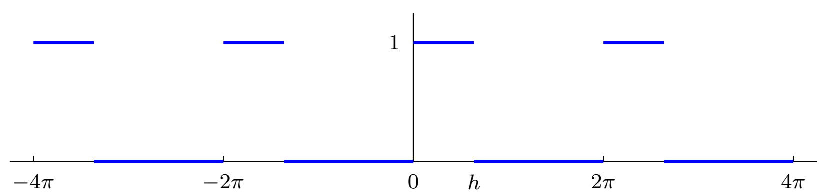


Fig. 10.10 A square pulse wave of period 2π.

► Solution. For this function, it is more convenient to compute the a_n and b_n using integration over the interval $[0,\, 2\pi]$ rather than the interval $[-\pi,\, \pi]$. Thus,

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(t)\,dt = \frac{1}{\pi}\int_0^h 1\,dt = \frac{h}{\pi},$$
$$a_n = \frac{1}{\pi}\int_0^h \cos nt\,dt = \frac{\sin nh}{\pi n}, \quad \text{and}$$
$$b_n = \frac{1}{\pi}\int_0^h \sin nt\,dt = \frac{1-\cos nh}{\pi n},$$

and the Fourier series is

$$f(t) \sim \frac{h}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{\sin nh}{n}\cos nt + \frac{1-\cos nh}{n}\sin nt\right).$$

◀

If the square pulse wave $f(t)$ is divided by h, then one obtains a function whose graph is a series of tall thin rectangles of height $1/h$ and base h, so that each of the rectangles with the bases starting at $2n\pi$ has area 1, as in Figure 10.11. Now consider the limiting case where h approaches 0. The

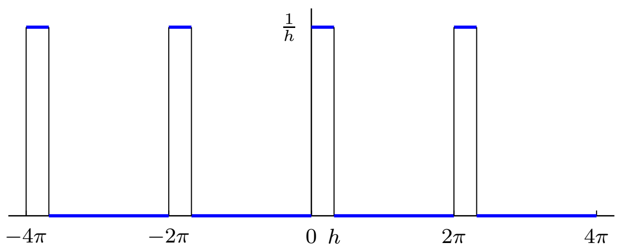


Fig. 10.11 A square unit pulse wave of period 2π.

graph becomes a series of infinite height spikes of width 0 as in Figure 10.12. This looks like an infinite sum of Dirac delta functions, which is the regular delta function extended to be periodic of period 2π. That is,

$$\lim_{h\to 0}\frac{f(t)}{h} = \sum_{n=-\infty}^{\infty}\delta(t-2n\pi).$$

Now compute the Fourier coefficients a_n/h and b_n/h as h approaches 0.

$$\begin{aligned} a_n &= \frac{\sin nh}{\pi nh} \to \frac{1}{\pi} \quad \text{and} \\ b_n &= \frac{1-\cos nh}{\pi nh} \to 0 \text{ as } h \to 0 \end{aligned}$$

Also, $a_0/h = 1/\pi$. Thus, the 2π-periodic delta function has a Fourier series

$$\sum_{n=-\infty}^{\infty}\delta(t-2n\pi) \sim \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\cos nt. \tag{12}$$

It is worth pointing out that this is one series that definitely does not converge for all t. In fact it does not converge for $t = 0$. However, there is a sense of convergence in which convergence makes sense. We will discuss this in the next section.

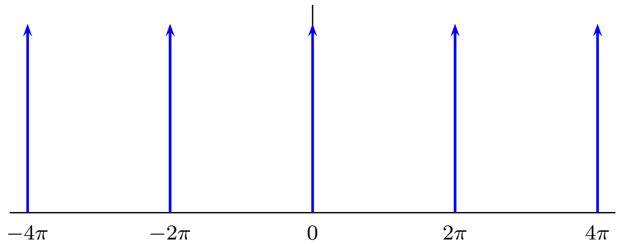


Fig. 10.12 The limiting square unit pulse wave of period 2π as $h \to 0$.

Example 8. Compute the Fourier series of the function $f(t) = \sin 2t + 5\cos 3t$.

▶ Solution. Since $f(t)$ is periodic of period 2π and is already given as a sum of sines and cosines, no further work is needed. The coefficients can be read off without any integration as $a_n = 0$ for $n \neq 3$ and $a_3 = 5$ while $b_n = 0$ for $n \neq 2$ and $b_2 = 1$. The same results would be obtained by integration using Euler's formulas.

Example 9. Compute the Fourier series of the function $f(t) = \sin^3 t$

▶ Solution. This can be handled by trig identities to reduce it to a finite sum of terms of the form $\sin nt$.

$$\begin{aligned}
\sin^3 t = \sin t \sin^2 t &= \frac{1}{2}\sin t(1 - \cos 2t) \\
&= \frac{1}{2}\sin t - \frac{1}{2}\sin t \cos 2t = \frac{1}{2}\sin t - \frac{1}{2}\big(\frac{1}{2}(\sin 3t + \sin(-t))\big) \\
&= \frac{3}{4}\sin t - \frac{1}{4}\sin 3t
\end{aligned}$$

The right hand side is the Fourier series of $\sin^3 t$.

Exercises

1–17. Find the Fourier series of the given periodic function.

1. $f(t) = \begin{cases} 0 & \text{if } -5 \le t < 0 \\ 3 & \text{if } 0 \le t < 5 \end{cases}; \qquad f(t+10) = f(t).$

2. $f(t) = \begin{cases} 2 & \text{if } -\pi \le t < 0 \\ -2 & \text{if } 0 \le t < \pi \end{cases}; \qquad f(t+2\pi) = f(t).$

3. $f(t) = \begin{cases} 4 & \text{if } -\pi \le t < 0 \\ -1 & \text{if } 0 \le t < \pi \end{cases}; \qquad f(t+2\pi) = f(t).$

4. $f(t) = t$, $0 \le t < 2$; $f(t+2) = f(t)$.
Hint: It may be useful to take advantage of Equation (10) in applying the Euler formulas.

5. $f(t) = t$, $-\pi \le t < \pi$; $f(t+2\pi) = f(t)$.

6. $f(t) = \begin{cases} 0 & 0 \le t < 1, \\ 1 & 1 \le t < 2, \\ 0 & 2 \le t < 3, \end{cases} \qquad f(t+3) = f(t).$

7. $f(t) = t^2$, $-2 \le t < 2$; $f(t+4) = f(t)$.

8. $f(t) = \begin{cases} 0 & \text{if } -1 \le t < 0 \\ t^2 & \text{if } 0 \le t < 1 \end{cases}; \qquad f(t+2) = f(t).$

9. $f(t) = \sin t$, $0 \le t < \pi$; $f(t+\pi) = f(t)$.

10. $f(t) = \begin{cases} 0 & \text{if } -\pi \le t < 0 \\ \sin t & \text{if } 0 \le t < \pi \end{cases}; \qquad f(t+2\pi) = f(t).$

11. $f(t) = \begin{cases} 1+t & \text{if } -1 \le t < 0 \\ 1-t & \text{if } 0 \le t < 1 \end{cases}; \qquad f(t+2) = f(t).$

12. $f(t) = \begin{cases} 1+t & \text{if } -1 \le t < 0 \\ -1+t & \text{if } 0 \le t < 1 \end{cases}; \qquad f(t+2) = f(t).$

13. $f(t) = \begin{cases} -t(\pi+t) & \text{if } -\pi \le t < 0 \\ t(\pi-t) & \text{if } 0 \le t < \pi \end{cases}; \qquad f(t+2\pi) = f(t).$

14. $f(t) = 3\cos^2 t$, $-\pi \le t < \pi$; $f(t+2\pi) = f(t)$

15. $f(t) = \sin \frac{t}{2}$, $-\pi \le t < \pi$; $f(t+2\pi) = f(t)$

16. $f(t) = \sin pt$, $-\pi \le t < \pi$; $f(t+2\pi) = f(t)$ (where p is not an integer)

17. $f(t) = e^t$; $-1 \le t < 1$; $f(t+2) = f(t)$

Then,

$$a_n = \int_{-1}^{1} e^t \cos n\pi t \, dt$$

$$
\begin{aligned}
&= \frac{1}{1+n^2\pi^2} e^t[\cos n\pi t + n\pi \sin n\pi t]\Big|_{-1}^{1} \\
&= \frac{1}{1+n^2\pi^2}[e^1 \cos n\pi - e^{-1}\cos(-n\pi)] \\
&= \frac{(e^1 - e^{-1})(-1)^n}{1+n^2\pi^2} = \frac{2(-1)^n \sinh(1)}{1+n^2\pi^2},
\end{aligned}
$$

and,

$$
\begin{aligned}
b_n &= \int_{-1}^{1} e^t \sin n\pi t\, dt \\
&= \frac{1}{1+n^2\pi^2} e^t[\sin n\pi t - n\pi \cos n\pi t]\Big|_{-1}^{1} \\
&= \frac{1}{1+n^2\pi^2}[e^1(-n\pi \cos n\pi) - e^{-1}(-n\pi\cos(-n\pi))] \\
&= \frac{(e^1 - e^{-1})(-n\pi)(-1)^n}{1+n^2\pi^2} = \frac{2(-1)^n(-n\pi)\sinh(1)}{1+n^2\pi^2}.
\end{aligned}
$$

Therefore, the Fourier series is

$$
f(t) \sim \sinh(1) + 2\sinh(1)\sum_{n=1}^{\infty} \frac{(-1)^n(\cos n\pi t - n\pi \sin n\pi t)}{1+n^2\pi^2}.
$$

10.3 Convergence of Fourier Series

If $f(t)$ is a periodic function of period $2L$, the Fourier series of $f(t)$ is

$$
f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi t}{L} + b_n \sin\frac{n\pi t}{L}\right), \tag{1}
$$

where the coefficients a_n and b_n are given by the Euler formulas

$$
a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi}{L}t\, dt, \qquad n = 0, 1, 2, 3, \dots \tag{2}
$$

$$
b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi}{L}t\, dt, \qquad n = 1, 2, 3, \dots \tag{3}
$$

The questions that we wish to consider are (1) under what conditions on the function $f(t)$ is it true that the series converges, and (2) when does it converge to the original function $f(t)$?

As with any infinite series, to say that $\sum_{n=1}^{\infty} c_n = C$ means that the sequence of partial sums $\sum_{n=1}^{m} c_n = S_m$ converges to C, that is,

$$\lim_{m\to\infty} S_m = C.$$

The partial sums of the Fourier series (1) are

$$S_m(t) = \frac{a_0}{2} + \sum_{n=1}^{m} \left(a_n \cos \frac{n\pi t}{L} + b_n \sin \frac{n\pi t}{L} \right). \tag{4}$$

The partial sum $S_m(t)$ is a finite linear combination of the trigonometric functions $\cos \frac{n\pi t}{L}$ and $\sin \frac{n\pi t}{L}$, each of which is periodic of period $2L$. Thus, $S_m(t)$ is also periodic of period $2L$, and if the partial sums converge to a function $g(t)$, then $g(t)$ must also be periodic of period $2L$.

Example 1. Consider the odd square wave function of period 2 and amplitude 1 from Example 2 of Section 10.2

$$f(t) = \begin{cases} -1 & -1 \le t < 0, \\ 1 & 0 \le t < 1, \end{cases}; \qquad f(t+2) = f(t). \tag{5}$$

The Fourier series of this function was found to be (letting $L = 1$)

$$\begin{aligned} f(t) &\sim \frac{4}{\pi} \left(\sin \pi t + \frac{1}{3} \sin 3\pi t + \frac{1}{5} \sin 5\pi t + \frac{1}{7} \sin 7\pi t + \cdots \right) \qquad (6) \\ &= \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin(2k+1)\pi t. \end{aligned}$$

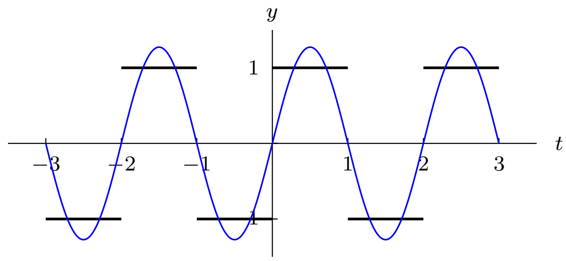
Letting $t = 0$ in this series gives

$$\frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin(2k+1)\pi 0 = \frac{4}{\pi} \sum_{k=0}^{\infty} 0 = 0.$$

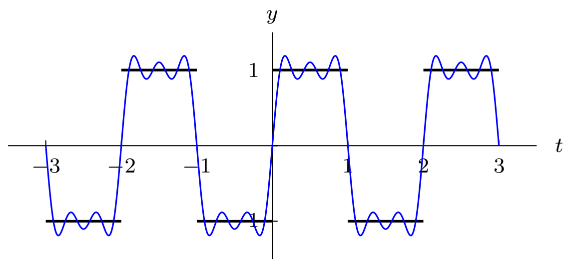
Since $f(0) = 1$, it follows that the Fourier series does not converge to $f(t)$ for all t. Several partial sums S_m are graphed along with the graph of $f(t)$ in Figure 10.13. It can be seen that the graph of $S_{15}(t)$ is very close to the graph of $f(t)$, except at the points of discontinuity, which are 0, ± 1, ± 2, ... This suggests that the Fourier series of $f(t)$ converges to $f(t)$ at all points except for where $f(t)$ has a discontinuity, which occurs whenever t is an integer n. For each integer n,

$$S_m(n) = \frac{4}{\pi} \sum_{k=0}^{[\frac{m}{2}]} \frac{1}{2k+1} \sin(2k+1)\pi n = \frac{4}{\pi} \sum_{k=0}^{[\frac{m}{2}]} 0 = 0.$$

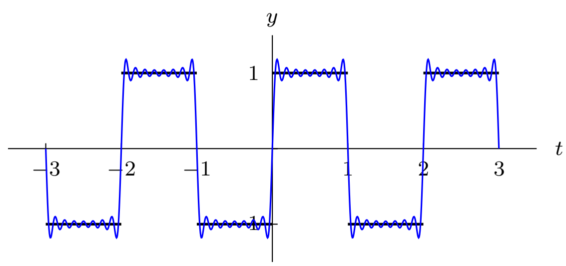
Hence the Fourier series converges to $f(t)$ whenever t is not an integer and it converges to 0, which is the midpoint of the jump of $f(t)$ at each integer.



$S_1(t) = \frac{4}{\pi}\sin \pi t$



$S_5(t) = \frac{4}{\pi}\left(\sin \pi t + \frac{1}{3}\sin 3\pi t + \frac{1}{5}\sin 5\pi t\right)$



$S_{15}(t) = \frac{4}{\pi}\sum_{k=0}^{7} \frac{1}{2k+1}\sin(2k+1)\pi t$

Fig. 10.13 Fourier series approximations S_m to the odd square wave of period 2 for $m = 1$, 5, and 15.

Similar results are true for a broad range of functions. We will describe the types of functions to which the convergence theorem applies and then state the convergence theorem.

Recall that a function $f(t)$ defined on an interval I has a **jump discontinuity** at a point $t = a$ if the left hand limit $f(a^-) = \lim_{t\to a^-} f(t)$ and the right hand limit $f(a^+) = \lim_{t\to a^+} f(t)$ both exist (as real numbers, not $\pm\infty$) and

$$f(a^+) \neq f(a^-).$$

The difference $f(a^+) - f(a^-)$ is frequently referred to as the **jump** in $f(t)$ at $t = a$. For example the odd square wave function of Example 10.2.2 has jump discontinuities at the points $\pm nL$ for $n = 0, 1, 2, \ldots$ and the sawtooth wave function of Example 10.2.5 has jump discontinuities at $\pm n$ for $n = 0, 1, 2, \ldots$. On the other hand, the function $g(t) = \tan t$ is a periodic function of period π that is discontinuous at the points $\frac{\pi}{2} + k\pi$ for all integers k, but the discontinuity is not a jump discontinuity since, for example

$$\lim_{t \to \frac{\pi}{2}^-} \tan t = +\infty, \text{ and } \lim_{t \to \frac{\pi}{2}^+} \tan t = -\infty.$$

A function $f(t)$ is **piecewise continuous** on a closed interval $[a, b]$ if $f(t)$ is continuous except for possibly finitely many jump discontinuities in the interval $[a, b]$. A function $f(t)$ is piecewise continuous for all t if it is piecewise continuous on every bounded interval. In particular, this means that there can be only finitely many discontinuities in any given bounded interval, and each of those must be a jump discontinuity. For convenience it will not be required that $f(t)$ be defined at the jump discontinuities, or $f(t)$ can be defined arbitrarily at the jump discontinuities. The square waves and sawtooth waves from Section 10.2 are typical examples of piecewise continuous periodic functions, while $\tan t$ is not piecewise continuous since the discontinuities are not jump discontinuities.

A function $f(t)$ is **piecewise smooth** if it is piecewise continuous and the derivative $f'(t)$ is also piecewise continuous. As with the convention on piecewise continuous, $f'(t)$ may not exist at finitely many points in any closed interval. All of the examples of periodic functions from Section 10.2 are piecewise smooth. The property of $f(t)$ being piecewise smooth and periodic is sufficient to guarantee that the Fourier series of $f(t)$ converges, and the sum can be computed. This is the content of the following theorem.

Theorem 2. ***(Convergence of Fourier Series)*** *Suppose that $f(t)$ is a periodic piecewise smooth function of period $2L$. Then the Fourier series* (1) *converges*

(a) *to the value $f(t)$ at each point t where f is continuous, and*
(b) *to the value $\frac{1}{2}[f(t^+) + f(t^-)]$ at each point where f is not continuous.*

Note that $\frac{1}{2}[f(t^+) + f(t^-)]$ is the *average* of the left-hand and right-hand limits of f at the point t. If f is continuous at t, then $f(t^+) = f(t) = f(t^-)$, so that

$$f(t) = \frac{f(t^+) + f(t^-)}{2}, \tag{7}$$

for any t where f is continuous. Hence Theorem 2 can be rephrased as follows:

The Fourier series of a piecewise smooth periodic function f converges for every t to the average of the left-hand and right-hand limits of f.

Example 3. Continuing with the odd square wave function of Example 1, it is clear from Equation (5) or from Figure 10.13 that if n is an even integer, then

$$\lim_{t \to n^+} f(t) = +1 \quad \text{and} \quad \lim_{t \to n^-} f(t) = -1,$$

while if n is an odd integer, then

$$\lim_{t \to n^+} f(t) = -1 \quad \text{and} \quad \lim_{t \to n^-} f(t) = +1.$$

Therefore, whenever n is an integer,

$$\frac{f(n^+) + f(n^-)}{2} = 0.$$

This is consistent with Theorem 2 since the Fourier series (6) converges to 0 whenever t is an integer n (because $\sin(2k+1)\pi n = 0$).

For any point t other than an integer, the function $f(t)$ is continuous, so Theorem 2 gives an equality

$$f(t) = \frac{4}{\pi}\left(\sin \pi t + \frac{1}{3}\sin 3\pi t + \frac{1}{5}\sin 5\pi t + \frac{1}{7}\sin 7\pi t + \cdots\right).$$

By redefining $f(t)$ at integers to be $f(n) = 0$ for integers n, then this equality holds for all t. Figure 10.14 shows the graph of the redefined function. Putting

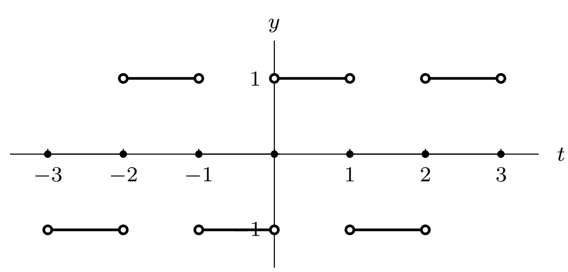


Fig. 10.14 The odd square wave of period 2 redefined so that the Fourier series converges to $f(t)$ for all t.

in the value of $f(t)$ for some values of t leads to a trigonometric identity

$$\frac{4}{\pi}\left(\sin \pi t + \frac{1}{3}\sin 3\pi t + \frac{1}{5}\sin 5\pi t + \frac{1}{7}\sin 7\pi t + \cdots\right) = f(t) = 1 \text{ if } 0 < t < 1.$$

Letting $t = 1/2$ then gives the series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

Example 4. Let f be periodic of period 2 and defined on the interval $0 \le t < 2$ by $f(t) = t^2$. Without computing the Fourier coefficients, give an explicit description of the sum of the Fourier series of f for all t.

► Solution. The function is defined in only one period so the periodic extension is defined by

$$f(t) = t^2 \quad \text{for } 0 \le t < 2; \qquad f(t+2) = f(t).$$

Both $f(t)$ and $f'(t)$ are continuous everywhere except at the even integers. Thus, the Fourier series converges to $f(t)$ everywhere except at the even integers. At the even integer $2m$ the left and right limits are

$$f(2m^-) = \lim_{t \to 2m^-} f(t) = \lim_{t \to 0^-} f(t) = \lim_{t \to 0^-} (t+2)^2 = 4, \text{ and}$$

$$f(2m^+) = \lim_{t \to 2m^+} f(t) = \lim_{t \to 0^+} f(t) = \lim_{t \to 0^+} t^2 = 0.$$

Therefore the sum of the Fourier series at $t = 2m$ is

$$\frac{f(2m^-) + f(2m^+)}{2} = \frac{4+0}{2} = 2.$$

The graph of the sum is shown in Figure 10.15. ◄

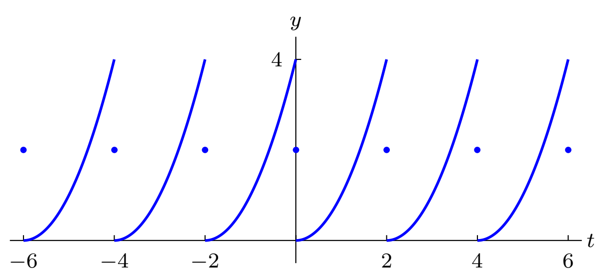


Fig. 10.15 The graph of sum of the Fourier series of the period 2 function $f(t)$ for Example 4.

Example 5. The 2π-periodic delta function $\sum_{n=-\infty}^{\infty} \delta(t - 2n\pi)$ has a Fourier series

$$\sum_{n=-\infty}^{\infty} \delta(t - 2n\pi) \sim \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \cos nt \qquad (8)$$

that was computed in Section 10.2. See Figure 10.12 for the graph of $\sum_{n=-\infty}^{\infty} \delta(t-2n\pi)$. The 2π-periodic delta function does not satisfy the conditions of the convergence theorem. Moreover, it is clear that the series does not converge since the individual terms do not approach 0. If $t=0$ the series becomes

$$\frac{1}{2\pi} + \frac{1}{\pi}[1+1+1+\cdots],$$

while, if $t=\pi$ the series is

$$\frac{1}{2\pi} + \frac{1}{\pi}[-1+1-1+\cdots].$$

Neither of these series are convergent. However, there is a sense of convergence in which the Fourier series of $\sum_{n=-\infty}^{\infty} \delta(t-2n\pi)$ "converges" to $\sum_{n=-\infty}^{\infty} \delta(t-2n\pi)$. To explore this phenomenon, we will explicitly compute the n^{th} partial sum $\varDelta_n(t)$ of the series (8):

$$\begin{aligned}\varDelta_n(t) &= \frac{1}{2\pi} + \frac{1}{\pi}\left[\cos t + \cos 2t + \cdots + \cos nt\right] \\ &= \frac{1}{2\pi}\left[1 + 2\cos t + 2\cos 2t + \cdots + 2\cos nt\right]\end{aligned}$$

Now use the Euler identity $2\cos\theta = e^{i\theta} + e^{-i\theta}$ to get

$$\varDelta_n(t) = \frac{1}{2\pi}\left[1 + e^{it} + e^{-it} + e^{i2t} + e^{-i2t} + \cdots + e^{int} + e^{-int}\right].$$

Letting $z = e^{it}$, this is seen to be a geometric series of ratio z that begins with the term z^{-n} and ends with the term z^n. Since the sum of a geometric series is known: $a + ar + ar^2 + \cdots + ar^m = a(1-r^{m+1})/(1-r)$ it follows that

$$\begin{aligned}\varDelta_n(t) &= \frac{1}{2\pi}\left[1 + e^{it} + e^{-it} + e^{i2t} + e^{-i2t} + \cdots + e^{int} + e^{-int}\right] \\ &= \frac{1}{2\pi}\left[z^{-n} + z^{-n+1} + \cdots + z^n\right] = \frac{1}{2\pi} z^{-n}\left[1 + z + \cdots + z^{2n}\right] \\ &= \frac{1}{2\pi} z^{-n}\frac{1-z^{2n+1}}{1-z} = \frac{1}{2\pi}\;\frac{z^{-n}-z^{n+1}}{1-z} \\ &= \frac{1}{2\pi}\;\frac{e^{-int}-e^{i(n+1)t}}{1-e^{it}} = \frac{1}{2\pi}\;\frac{e^{i(n+1)t}-e^{-int}}{e^{it}-1}\frac{e^{-it/2}}{e^{-it/2}} \\ &= \frac{1}{2\pi}\;\frac{e^{i\left(n+\frac{1}{2}\right)t} - e^{-i\left(n+\frac{1}{2}\right)t}}{e^{it/2}-e^{-it/2}} \\ &= \frac{1}{2\pi}\;\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}.\end{aligned}$$

For those values of t where $\sin \frac{1}{2}t = 0$, the value of $\Delta_n(t)$ is the limiting value. The graphs of $\Delta_n(t)$ for $n = 5$ and $n = 15$ over 1 period $-\pi \le t \le \pi]$ are shown in Figure 10.16. As n increases the central hump increases in height and becomes narrower, while away from 0 the graph is a rapid oscillation of small amplitude. Also, the area enclosed by $\Delta_n(t)$ in one period $-\pi \le t \le \pi$ is always 1. This is easily seen from the expression of $\Delta_n(t)$ as a sum of cosine terms:

$$\int_{-\pi}^{\pi} \Delta_n(t)\, dt = \int_{-\pi}^{\pi} \frac{1}{2\pi} \left[1 + 2\cos t + 2\cos 2t + \cdots + 2\cos nt\right]\, dt = 1$$

since $\int_{-\pi}^{\pi} \cos mt\, dt = 0$ for all nonzero integers m. Most of the area is concentrated in the first hump since the rapid oscillation cancels out the contributions above and below the t axis away from this hump.

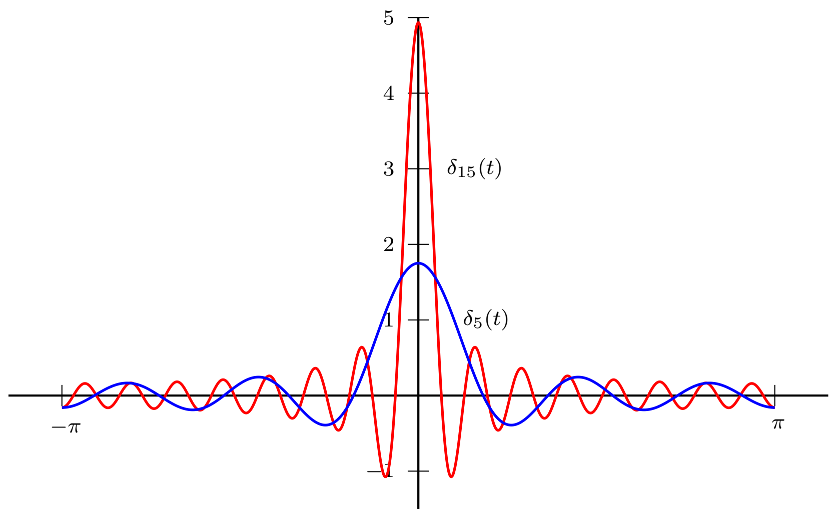


Fig. 10.16 The graphs of the partial sums $\delta_n(t)$ for $n = 5$ and $n = 15$.

To see the sense in which $\Delta_n(t)$ converges to $\sum_{n=-\infty}^{\infty} \delta(t - 2n\pi)$, recall that $\delta(t)$ is described via integration by the property that $\delta(t) = 0$ if $t \ne 0$, and $\int f(t)\delta(t)\, dt = f(0)$ for all test functions $f(t)$. Since we are working with periodic functions, consider the integrals over 1 period, in our case $[-\pi, \pi]$. Thus let $f(t)$ be a smooth function and expand it in a cosine series $f(t) = \sum a_m \cos mt$. Multiply by $\Delta_n(t)$ and integrate, using the orthogonality conditions to get

$$\int_{-\pi}^{\pi} \Delta_n(t) f(t)\, dt = a_0 + \cdots + a_n. \tag{9}$$

As $n \to \infty$ this sum converges to $f(0)$. Thus

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} \Delta_n(t)f(t)\,dt = f(0) = \int_{-\pi}^{\pi} \delta(t)f(t)\,dt.$$

It is in this weak sense that $\Delta_n(t) \to \sum_{n=-\infty}^{\infty} \delta(t-2n\pi)$. Namely, the effect under integration is the same.

Exercises

1–9. In each exercise, (a) Sketch the graph of $f(t)$ over at least 3 periods. (b) Determine the points t at which the Fourier series of $f(t)$ converges to $f(t)$. (c) At each point t of discontinuity, give the value of $f(t)$ and the value to which the Fourier series converges.

1. $f(t) = \begin{cases} 3 & \text{if } 0 \le t < 2 \\ -1 & \text{if } 2 \le t < 4 \end{cases}; \qquad f(t+4) = f(t).$

2. $f(t) = t$, $-1 \le t \le 1$, $f(t+2) = f(t)$.

3. $f(t) = t$, $0 \le t \le 2$, $f(t+2) = f(t)$.

4. $f(t) = |t|$, $-1 \le t \le 1$, $f(t+2) = f(t)$.

5. $f(t) = t^2$, $-2 \le t \le 2$, $f(t+4) = f(t)$.

6. $f(t) = 2t-1$, $-1 \le t < 1$, $f(t+2) = f(t)$.

7. $f(t) = \begin{cases} 2 & \text{if } -2 \le t < 0 \\ t & \text{if } 0 \le t < 2 \end{cases}; \qquad f(t+4) = f(t).$

8. $f(t) = \cos(t/2)$, $0 \le t < \pi$, $f(t+\pi) = f(t)$.

9. $f(t) = |\cos \pi t|$, $0 \le t < 1$, $f(t+1) = f(t)$.

10. Verify that the following function is not piecewise smooth.

$$f(t) = \begin{cases} \sqrt{t} & \text{if } 0 \le t < 1 \\ 0 & \text{if } -1 \le t < 0. \end{cases}$$

11. Verify that

$$\sin t - \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t - \frac{1}{4}\sin 4t + \cdots = \frac{t}{2}, \qquad \text{for } -\pi < t < \pi.$$

Explain how this gives the summation

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

12. Use the Fourier series of the function $f(t) = |t|$, $-\pi < t < \pi$ (see Example 4 of Section 10.2) to compute the sum of the series

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots .$$

13. Verify that

$$\frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi t = t^2, \qquad \text{for } -1 \le t \le 1.$$

14. Using the Fourier series in problem 13, find the sum of the following series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

15. Compute the Fourier series for the function $f(t) = t^4$, $-\pi \le t \le \pi$; $f(t + 2\pi) = f(t)$ for all t. Using this result, verify the summations

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{720}.$$

10.4 Fourier Sine Series and Fourier Cosine Series

In Section 10.2 we observed (see the discussion following Example 5) that the Fourier series of an even function will only have cosine terms, while the Fourier series of an odd function will only have sine terms. We will take advantage of this feature to obtain new types of Fourier series representations that are valid for functions defined on some interval $0 \le t \le L$. In practice, we will frequently want to represent $f(t)$ by a Fourier series that involves only cosine terms or only sine terms. To do this, we will first extend $f(t)$ to the interval $-L \le t \le 0$. We may then extend $f(t)$ to all of the real line by assuming that $f(t)$ is $2L$ periodic. That is, assume $f(t + 2L) = f(t)$. We will extend $f(t)$ in such a way as to obtain either an even function or an odd function.

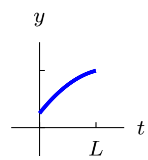
If $f(t)$ is defined only for $0 \le t \le L$, then the **even $2L$-periodic extension** of $f(t)$ is the function $f_e(t)$ defined by

$$f_e(t) = \begin{cases} f(t) & \text{if } 0 \le t \le L, \\ f(-t) & \text{if } -L \le t < 0, \end{cases}; \quad f_e(t+2L) = f_e(t). \tag{1}$$

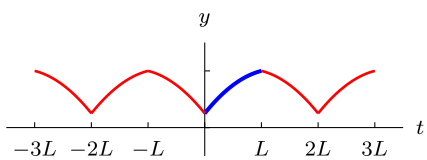
The **odd** $2L$**-periodic extension** of $f(t)$ is the function $f_o(t)$ defined by

$$f_e(t) = \begin{cases} f(t) & \text{if } 0 \le t \le L, \\ -f(-t) & \text{if } -L \le t < 0, \end{cases}; \quad f_o(t+2L) = f_o(t). \tag{2}$$

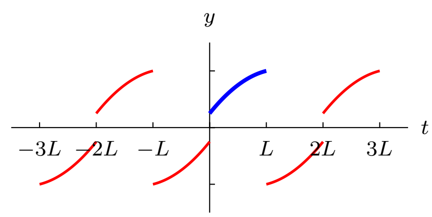
These two extensions are illustrated for a particular function $f(t)$ in Figure 10.17. Note that both $f_e(t)$ and $f_o(t)$ agree with $f(t)$ on the interval $0 < t < L$.



The graph of $f(t)$ on the interval $0 \le t \le L$.



The graph of $f_e(t)$, the even $2L$-periodic extension of $f(t)$.



The graph of $f_o(t)$, the odd $2L$-periodic extension of $f(t)$.

Fig. 10.17

Since $f_e(t)$ is an even $2L$-periodic function, then $f_e(t)$ has a Fourier series expansion

$$f_e(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi t}{L} + b_n \sin \frac{n\pi t}{L} \right)$$

where the coefficients are given by Euler's formulas (Equations (6)-(8) of Section 10.2)

$$a_0 = \frac{1}{L} \int_{-L}^{L} f_e(t)\, dt$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f_e(t)\cos\frac{n\pi}{L}t\,dt, \qquad n = 1, 2, 3, \ldots$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f_e(t)\sin\frac{n\pi}{L}t\,dt, \qquad n = 1, 2, 3, \ldots.$$

Since $f_e(t)$ is an even function, then $f_e(t)\sin\frac{n\pi}{L}t$ is odd, so the integral defining b_n is 0 (see Proposition 5 of Section 10.1). Thus, $b_n = 0$ and we get the Fourier series expansion

$$f_e(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L},$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f_e(t)\cos\frac{n\pi t}{L}\,dt, \qquad \text{for } n \geq 0.$$

Since the integrand is even and $f_e(t) = f(t)$ for $0 \leq t \leq L$, it follows that

$$a_n = \frac{2}{L}\int_{0}^{L} f(t)\cos\frac{n\pi t}{L}\,dt, \qquad \text{for } n \geq 0. \tag{3}$$

If we assume that $f(t)$ is piecewise smooth on the interval $0 \leq t \leq L$, then the even $2L$-periodic extension $f_e(t)$ is also piecewise smooth, and the convergence theorem (Theorem 2 of Section 10.3) shows that the Fourier series of $f_e(t)$ converges to the average of the left-hand and right-hand limits of f_e at each t. By restricting to the interval $0 \leq t \leq L$, we obtain the series expansion

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L}, \qquad \text{for } 0 \leq t \leq L, \tag{4}$$

where the coefficients are given by (3) and this series converges to the average of the left-hand and right-hand limits of f at each t in the interval $[0, L]$. The series in (4) is called the **Fourier cosine series** for f on the interval $[0, L]$.

Similarly, by using the Fourier series expansion of the odd $2L$-periodic extension of $f(t)$, and restricting to the interval $0 \leq t \leq L$, we obtain the series expansion

$$f(t) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L}, \qquad \text{for } 0 \leq t \leq L, \tag{5}$$

where the coefficients are given by

$$b_n = \frac{2}{L}\int_{0}^{L} f(t)\sin\frac{n\pi t}{L}\,dt, \qquad \text{for } n \geq 1, \tag{6}$$

and this series converges to the average of the left-hand and right-hand limits of f at each t in the interval $[0,\, L]$. The series in (5) is called the **Fourier sine series** for f on the interval $[0,\, L]$.

Example 1. Let $f(t) = \pi - t$ for $0 \le t \le \pi$. Compute both the Fourier cosine series and the Fourier sine series for the function $f(t)$. Sketch the graph of the function to which the Fourier cosine series converges and the graph of the function to which to Fourier sine series converges.

▶ Solution. In this case, the length of the interval is $L = \pi$ and from (4), the Fourier cosine series of $f(t)$ is

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt,$$

with

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(t)\,dt = \frac{2}{\pi}\int_0^{\pi} (\pi - t)\,dt = \pi,$$

and for $n \ge 1$,

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} f(t)\cos nt\,dt = \frac{2}{\pi}\int_0^{\pi}(\pi - t)\cos nt\,dt \\
&= \frac{2}{n}\sin nt\Big|_0^{\pi} - \frac{2}{\pi}\int_0^{\pi} t\cos nt\,dt \\
&= -\frac{2}{\pi}\left[\frac{t}{n}\sin nt\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin nt\,dt\right] \\
&= \frac{2}{\pi n^2}(-\cos nt)\Big|_0^{\pi} = \frac{2}{\pi n^2}(-\cos n\pi + 1) \\
&= \frac{2}{\pi n^2}(1 - (-1)^n) \\
&= \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{4}{\pi n^2} & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

Thus, the Fourier cosine series of $f(t)$ is

$$f(t) = \pi - t = \frac{\pi}{2} + \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \cdots \frac{1}{(2k-1)^2}\cos(2k-1)t + \cdots\right).$$

This series converges to $f(t)$ for $0 \le t \le \pi$ and to the even 2π-periodic extension $f_e(t)$ otherwise, since $f_e(t)$ is continuous, as can be seen from the graph of $f_e(t)$ which is given in Figure 10.18.

Similarly, from (5), the Fourier sine series of $f(t)$ is

$$f(t) \sim \sum_{n=1}^{\infty} b_n \sin nt$$

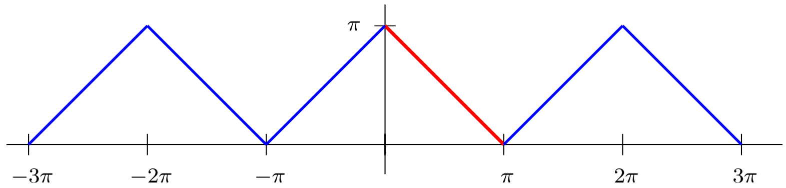


Fig. 10.18 The even 2π-periodic extension $f_e(t)$ of the function $f(t) = \pi - t$ defined on $0 \le t \le \pi$.

with (for $n \ge 1$)

$$
\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^\pi f(t)\sin nt\,dt = \frac{2}{\pi}\int_0^\pi (\pi - t)\sin nt\,dt \\
&= -\frac{2}{n}\cos nt\Big|_0^\pi - \frac{2}{\pi}\int_0^\pi t\sin nt\,dt \\
&= -\frac{2}{n}\cos nt\Big|_0^\pi - \frac{2}{\pi}\left[-\frac{t}{n}\cos nt\Big|_0^\pi + \frac{1}{n}\int_0^\pi \cos nt\,dt\right] \\
&= \frac{2}{n}
\end{aligned}
$$

Thus, the Fourier sine series of $f(t)$ is

$$
f(t) = \pi - t = 2\sum_{n=1}^\infty \frac{\sin nt}{n}.
$$

This series converges to $f(t)$ for $0 < t < \pi$ and to the odd 2π-periodic extension $f_o(t)$ otherwise, except for the jump discontinuities of $f_o(t)$, which occur at the multiples of $n\pi$, as can be seen from the graph of $f_o(t)$ which is given in Figure 10.19. At $t = n\pi$, the series converges to 0. ◀

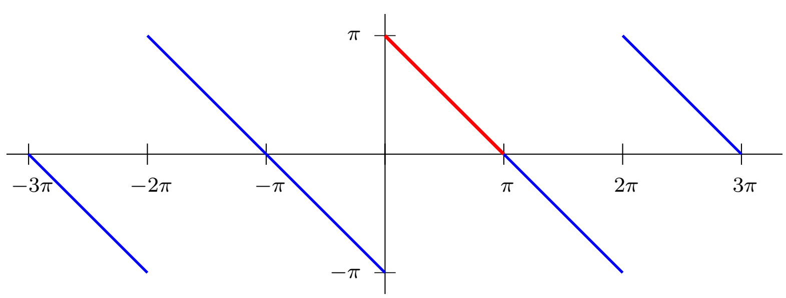


Fig. 10.19 The odd 2π-periodic extension $f_e(t)$ of the function $f(t) = \pi - t$ defined on $0 \le t \le \pi$.

Exercises

1–11. A function $f(t)$ is defined on an interval $0 < t < L$. Find the Fourier cosine and sine series of f and sketch the graphs of the two extensions of f to which these two series converge.

1. $f(t) = 1$, $0 < t < L$

2. $f(t) = t$, $0 < t < 1$

3. $f(t) = t$, $0 < t < 2$

4. $f(t) = t^2$, $0 < t < 1$

5. $f(t) = \begin{cases} 1 & \text{if } 0 < t < \pi/2 \\ 0 & \text{if } \pi/2 < t < \pi \end{cases}$

6. $f(t) = \begin{cases} t & \text{if } 0 < t \leq 1, \\ 0 & \text{if } 1 < t < 2 \end{cases}$

7. $f(t) = t - t^2$, $0 < t < 1$

8. $f(t) = \sin t$, $0 < t < \pi$

9. $f(t) = \cos t$, $0 < t < \pi$

10. $f(t) = e^t$, $0 < t < 1$

11. $f(t) = 1 - (2/L)t$, $0 < t < L$

10.5 Operations on Fourier Series

In applications it is convenient to know how to compute the Fourier series of function obtained from other functions by standard operations such addition, scalar multiplication, differentiation, and integration. We will consider conditions under which these operations can be used to facilitate Fourier series calculations. First we consider linearity.

Theorem 1. *If f and g are $2L$-periodic functions with Fourier series*

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi t}{L} + b_n \sin \frac{n\pi t}{L} \right)$$

and

$$g(t) \sim \frac{c_0}{2} + \sum_{n=1}^{\infty} \left(c_n \cos \frac{n\pi t}{L} + d_n \sin \frac{n\pi t}{L} \right),$$

then for any constants α and β the function defined by $h(t) = \alpha f(t) + \beta g(t)$ is $2L$-periodic with Fourier series

$$h(t) \sim \frac{\alpha a_0 + \beta c_0}{2} + \sum_{n=1}^{\infty} \left((\alpha a_n + \beta c_n) \cos \frac{n\pi t}{L} + (\alpha b_n + \beta d_n) \sin \frac{n\pi t}{L} \right). \quad (1)$$

This result follows immediately from the Euler formulas for the Fourier coefficients and from the linearity of the definite integral.

Example 2. Compute the Fourier series of the following periodic functions.

1. $h_1(t) = t + |t|$, $\pi \le t < \pi$, $h_1(t + 2\pi) = h(t)$.
2. $h_2(t) = 1 + 2t$, $-1 \le t < 1$, $h_2(t + 2) = h(t)$.

► Solution. The following Fourier series were calculated in Examples 4 and 5 of Section 10.2:

$$|t| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\cos(2k+1)t}{(2k+1)^2}$$

and

$$t = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi}{L} t.$$

Since h_1 is 2π-periodic, we take $L = \pi$ in the Fourier series for t over the interval $L \le t \le L$. By the linearity theorem we then obtain

$$h_1(t) = t + |t| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\cos(2k+1)t}{(2k+1)^2} + 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nt.$$

Since $h_1(t)$ is piecewise smooth, the equality is valid here, with the convention that the function is redefined to be $(h_1(t^+) + h_1(t^-))/2$ at each point t where the function is discontinuous. The graph of $h_1(t)$ is similar to that of the half sawtooth wave from Figure 10.9, where the only difference is that the period of $h_1(t)$ is 2π, rather than 4.

The function $h_2(t) = 1 + 2t$ is a sum of an even function 1 and an odd function $2t$. Since the constant function 1 is it's own Fourier series (that is, $a_0 = 2$, $a_n = 0 = b_n$ for $n \ge 1$, then letting $L = 1$ in the Fourier series for t, we get

$$h_2(t) = 1 + 2t = 1 + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\pi t.$$

◄

Differentiation of Fourier Series

We next consider the term-by-term differentiation of a Fourier series. This will be needed for the solution of some differential equations by substituting the Fourier series of an unknown function into the differential equation. Some care is needed since the term-by-term differentiation of a series of functions is not always valid. The following result gives sufficient conditions for term-by-term differentiation of Fourier series. Note first that if $f(t)$ is a periodic function of period p, then the derivative $f'(t)$ (where the derivative exists) is also periodic with period p. This follows immediately from the chain rule: Since $f(t+p) = f(t)$ for all t,

$$f'(t+p) = \frac{d}{dt}f(t+p) = \frac{d}{dt}f(t) = f'(t).$$

Theorem 3. *Suppose that f is a $2L$-periodic function that is continuous for all t, and suppose that the derivative f' is piecewise smooth for all t. If the Fourier series of f is*

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi t}{L} + b_n \sin\frac{n\pi t}{L}\right), \tag{2}$$

then the Fourier series of f' is the series

$$f'(t) \sim \sum_{n=1}^{\infty}\left(-\frac{n\pi}{L}a_n \sin\frac{n\pi t}{L} + \frac{n\pi}{L}b_n \cos\frac{n\pi t}{L}\right) \tag{3}$$

obtained by term-by-term differentiation of (2). *Moreover, the differentiated series* (3) *converges to $f'(t)$ for all t for which $f'(t)$ exists.*

Proof. Since f' is assumed to be $2L$-periodic and piecewise smooth, the convergence theorem (Theorem 2 of Section 10.3) shows that the Fourier series of $f'(t)$ converges to the average of the left-hand and right-hand limits of f' at each t. That is,

$$f'(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi t}{L} + B_n \sin\frac{n\pi t}{L}\right), \tag{4}$$

where the coefficients A_n and B_n are given by the Euler formulas

$$A_n = \frac{1}{L}\int_{-L}^{L} f'(t)\cos\frac{n\pi t}{L}\,dt \quad \text{and} \quad B_n = \frac{1}{L}\int_{-L}^{L} f'(t)\sin\frac{n\pi t}{L}\,dt.$$

To prove the theorem, it is sufficient to show that the series in Equations (3) and (4) are the same. From the Euler formulas for the Fourier coefficients of f',

$$A_0 = \frac{1}{L}\int_{-L}^{L} f'(t)\,dt = \frac{1}{L}\, f(t)|_{-L}^{L} = f(L) - f(-L) = 0$$

since f is continuous and $2L$-periodic. For $n \geq 1$, the Euler formulas for both f' and f, and integration by parts give

$$\begin{aligned} A_n &= \frac{1}{L}\int_{-L}^{L} f'(t)\cos\frac{n\pi t}{L}\,dt \\ &= \frac{1}{L}\, f(t)\cos\frac{n\pi t}{L}\bigg|_{-L}^{L} + \frac{n\pi}{L}\cdot\frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt \\ &= \frac{n\pi}{L} b_n, \end{aligned}$$

since $f(-L) = f(L)$, and

$$\begin{aligned} B_n &= \frac{1}{L}\int_{-L}^{L} f'(t)\sin\frac{n\pi t}{L}\,dt \\ &= \frac{1}{L}\, f(t)\sin\frac{n\pi t}{L}\bigg|_{-L}^{L} - \frac{n\pi}{L}\cdot\frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt \\ &= -\frac{n\pi}{L} a_n. \end{aligned}$$

Therefore, the series in Equations (3) and (4) are the same. □

Example 4. The even triangular wave function of period 2π given by

$$f(t) = \begin{cases} -t & -\pi \leq t < 0, \\ t & 0 \leq t < \pi, \end{cases}; \qquad f(t + 2\pi) = f(t),$$

whose graph is shown in Figure 10.7, is continuous for all t, and

$$f'(t) = \begin{cases} -1 & -\pi \leq t < 0, \\ 1 & 0 < t < \pi, \end{cases}; \qquad f'(t + 2\pi) = f(t),$$

is piecewise smooth. In fact f' is an odd square wave of period 2π as in Figure 10.5. Therefore, f satisfies the hypotheses of Theorem 3. Thus, the Fourier series of f:

$$f(t) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos t}{1^2} + \frac{\cos 3t}{3^2} + \frac{\cos 5t}{5^2} + \frac{\cos 7t}{7^2} + \cdots\right)$$

(computed in Example 4 of Section 10.2) can be differentiated term by term. The result is

$$f'(t) = \frac{4}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \frac{1}{7}\sin 7t + \cdots\right),$$

which is the Fourier series of the odd square wave of period 2π computed in Eq (11) of Section 10.2. ◀

Example 5. The sawtooth wave function of period $2L$ given by

$$f(t) = t \quad \text{for } -L \le t < L; \qquad f(t+2L) = f(t).$$

(see Example 5 of Section 10.2) has Fourier series

$$\begin{aligned} f(t) &\sim \frac{2L}{\pi}\left(\sin\frac{\pi}{L}t - \frac{1}{2}\sin\frac{2\pi}{L}t + \frac{1}{3}\sin\frac{3\pi}{L}t - \frac{1}{4}\sin\frac{4\pi}{L}t + \cdots\right) \\ &= \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi}{L}t. \end{aligned}$$

Since the function t is continuous, it might seem that this function satisfies the hypotheses of Theorem 3. However, looking at the graph of the function (see Figure 10.8) shows that the function is discontinuous at the points $t = \pm L$, $t = \pm 3L, \ldots$. Term-by-term differentiation of the Fourier series of f gives

$$\begin{aligned} &2\left(\cos\frac{\pi}{L}t - \cos\frac{2\pi}{L}t + \cos\frac{3\pi}{L}t - \cos\frac{4\pi}{L}t + \cdots\right) \\ &= 2\sum_{n=1}^{\infty}(-1)^{n+1}\cos\frac{n\pi}{L}t. \end{aligned}$$

This series does not converge for any point t. However, it is possible to make some sense out of this series using the Dirac delta function. ◀

Integration of Fourier Series

We now consider the termwise integration of the Fourier series of a piecewise continuous function. Start with a piecewise continuous $2L$-periodic function f and define an antiderivative of f by

$$g(t) = \int_{-L}^{t} f(x)\,dx. \tag{5}$$

Then g is a continuous function. It is not automatic that g is periodic. However, if

$$g(L) = \int_{-L}^{L} f(x)\,dx = 0, \tag{6}$$

it follows that

$$g(t+2L) = \int_{-L}^{t+2L} f(x)\,dx = \int_{-L}^{L} f(x)\,dx + \int_{L}^{t+2L} f(x)\,dx$$

$$= \int_{L}^{t+2L} f(x-2L)\,dx \quad \text{since } f \text{ is } 2L\text{-periodic}$$
$$= \int_{-L}^{t} f(u)\,du \quad \text{using the change of variables } u = x-2L$$
$$= g(t),$$

so that g is periodic of period $2L$. Therefore we have verified the following result.

Theorem 6. *Suppose that f is a piecewise continuous $2L$-periodic function. Then the antiderivative g of f defined by (5) is a continuous piecewise smooth $2L$-periodic function provided that (6) holds.*

Now we can state the main result on termwise integration of Fourier series.

Theorem 7. *Suppose that f is piecewise continuous and $2L$-periodic with Fourier series*

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi t}{L} + b_n \sin \frac{n\pi t}{L} \right). \tag{7}$$

If $g(t) = \int_{-L}^{t} f(x)\,dx$ and $g(L) = \int_{-L}^{L} f(x)\,dx = 0$, which is equivalent to $a_0 = 0$, then the Fourier series (7) can be integrated term-by-term to give the Fourier series

$$g(t) = \int_{-L}^{t} f(x)\,dx \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(\frac{L}{n\pi} a_n \sin \frac{n\pi t}{L} - \frac{L}{n\pi} b_n \cos \frac{n\pi t}{L} \right) \tag{8}$$

where

$$A_0 = -\frac{1}{L} \int_{-L}^{L} t f(t)\,dt. \tag{9}$$

The integrated series (8) converges to $g(t) = \int_{-L}^{t} f(x)\,dx$ for all t.

Proof. Since f is assumed to be $2L$-periodic and piecewise continuous and $a_0 = 0$, Theorem 6 and the convergence theorem shows that the Fourier series of $g(t)$ converges to $g(t)$ at each t. That is,

$$g(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi t}{L} + B_n \sin \frac{n\pi t}{L} \right), \tag{10}$$

where the coefficients A_n and B_n are given by the Euler formulas

$$A_n = \frac{1}{L} \int_{-L}^{L} g(t) \cos \frac{n\pi t}{L}\,dt \quad \text{and} \quad B_n = \frac{1}{L} \int_{-L}^{L} g(t) \sin \frac{n\pi t}{L}\,dt.$$

For $n \geq 1$ apply integration by parts using $u = g(t)$, $dv = \cos \frac{n\pi}{L} t\,dt$ so that $du = g'(t)\,dt = f(t)\,dt$, $v = \frac{L}{n\pi} \sin \frac{n\pi}{L} t$. This gives

$$\begin{aligned}A_n &= \frac{1}{L}\int_{-L}^{L} g(t)\cos\frac{n\pi t}{L}\,dt \\ &= g(t)\frac{1}{n\pi}\sin\frac{n\pi t}{L}\Big|_{-L}^{L} - \frac{1}{n\pi}\int_{-L}^{L} g'(t)\sin\frac{n\pi t}{L}\,dt \\ &= -\frac{L}{n\pi}\left(\frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt\right).\end{aligned}$$

Hence, $A_n = -\frac{L}{n\pi}b_n$ where b_n is the corresponding Fourier coefficient for f. Similarly, $B_n = \frac{L}{n\pi}a_n$ and using integration by parts with $u = g(t)$, $dv = dt$ so that $du = g'(t)\,dt = f(t)\,dt$, $v = t$,

$$A_0 = \frac{1}{L}\int_{-L}^{L} g(t)\,dt = \frac{1}{L}tg(t)\Big|_{-L}^{L} - \frac{1}{L}\int_{-L}^{L} tf(t)\,dt = -\frac{1}{L}\int_{-L}^{L} tf(t)\,dt.$$

□

Example 8. Let $f(t)$ be the odd square wave of period $2L$:

$$f(t) = \begin{cases} -1 & -L \le t < 0, \\ 1 & 0 \le t < L, \end{cases}; \qquad f(t+2L) = f(t),$$

which has Fourier series

$$f(t) \sim \frac{4}{\pi}\left(\sin\frac{\pi}{L}t + \frac{1}{3}\sin\frac{3\pi}{L}t + \frac{1}{5}\sin\frac{5\pi}{L}t + \frac{1}{7}\sin\frac{7\pi}{L}t + \cdots\right)$$

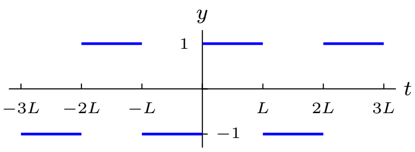
that was computed in Eq (11) of Section 10.2. For $-L \le t < 0$, $g(t) = \int_{-L}^{t} f(x)\,dx = \int_{-L}^{t}(-1)\,dx = -t - L$ and for $0 \le t < L$, $g(t) = \int_{-L}^{t} f(x)\,dx = \int_{-L}^{0}(-1)\,dx + \int_{0}^{t} 1\,dx = t - L$. Thus, $g(t)$ is the even triangular wave function of period $2L$ shifted down by L. That is, in the interval $-L \le t \le L$

$$g(t) = \int_{-L}^{t} f(x)\,dx = |t| - L$$

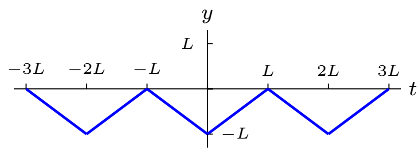
with $2L$ periodic extension to the rest of the real line. See Figure 10.20.

Theorem 7 then gives, for $-L \le t \le L$,

$$\begin{aligned}g(t) = |t| - L &= \frac{A_0}{2} + \frac{4}{\pi}\left(-\frac{L}{\pi}\cos\frac{\pi}{L}t - \frac{L}{9\pi}\cos\frac{3\pi}{L}t - \frac{L}{25\pi}\cos\frac{5\pi}{L}t - \cdots\right) \\ &= \frac{A_0}{2} - \frac{4L}{\pi^2}\left(\cos\frac{\pi}{L}t + \frac{1}{9}\cos\frac{3\pi}{L}t + \frac{1}{25}\cos\frac{5\pi}{L}t + \cdots\right)\end{aligned}$$



(a) $f(t) = \begin{cases} -1 & -L \le t < 0, \\ 1 & 0 \le t < L. \end{cases}$



(b) $g(t) = \int_{-L}^{t} f(x)\,dx$

Fig. 10.20

$$= \frac{A_0}{2} - \frac{4L}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos\frac{(2k-1)\pi}{L}t.$$

This can also be written as

$$|t| = \left(L + \frac{A_0}{2}\right) - \frac{4L}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos\frac{(2k-1)\pi}{L}t.$$

The constant term $L + A_0/2$ can be determined from Eq. (9):

$$L + \frac{A_0}{2} = L - \frac{1}{2L}\int_{-L}^{L} t f(t)\,dt = L - \frac{1}{2L}\int_{-L}^{L} |t|\;dt = L - \frac{L}{2} = \frac{L}{2}.$$

The constant term $L + A_0/2$ can also be determined as one-half of the zero Fourier coefficient of $|t|$:

$$L + \frac{A_0}{2} = \frac{1}{2L}\int_{-L}^{L} |t|\;dt = \frac{L}{2}.$$

◀

Exercises

1. Let the 2π-periodic function $f_1(t)$ and $f_2(t)$ be defined on $-\pi \le t < \pi$ by

$$\begin{aligned} f_1(t) &= 0, \quad -\pi \le t < 0; \quad f_1(t) = 1, \quad 0 \le t < \pi; \\ f_2(t) &= 0, \quad -\pi \le t < 0; \quad f_2(t) = t, \quad 0 \le t < \pi. \end{aligned}$$

Then the Fourier series of these functions are

$$f_1(t) \sim \frac{1}{2} + \frac{2}{\pi}\sum_{n=\text{odd}}\frac{\sin nt}{n},$$

$$f_2(t) \sim \frac{\pi}{4} - \frac{2}{\pi} \sum_{n=\text{odd}} \frac{\cos nt}{n^2} + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nt}{n}.$$

Without further integration, find the Fourier series for the following 2π-periodic functions:

(a) $f_3(t) = 1, \quad -\pi \le t < 0; \quad f_3(t) = 0, \quad 0 \le t < \pi;$
(b) $f_4(t) = t, \quad -\pi \le t < 0; \quad f_4(t) = 0, \quad 0 \le t < \pi;$
(c) $f_5(t) = 1, \quad -\pi \le t < 0; \quad f_5(t) = t, \quad 0 \le t < \pi;$
(d) $f_6(t) = 2, \quad -\pi \le t < 0; \quad f_6(t) = 0, \quad 0 \le t < \pi;$
(e) $f_7(t) = 2, \quad -\pi \le t < 0; \quad f_7(t) = 3, \quad 0 \le t < \pi;$
(f) $f_8(t) = 1, \quad -\pi \le t < 0; f_8(t) = 1 + 2t, 0 \le t < \pi;$
(g) $f_9(t) = a + bt, -\pi \le t < 0; f_9(t) = c + dt, 0 \le t < \pi.$

2. Let $f_1(t) = t$ for $-\pi < t < \pi$ and $f_1(t + 2\pi) = f_1(t)$ so that the Fourier series expansion for $f_1(t)$ is

$$f_1(t) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nt.$$

Using this expansion and Theorem 7 compute the Fourier series of each of the following 2π-periodic functions.

(a) $f_2(t) = t^2$ for $-\pi < t < \pi$.
(b) $f_3(t) = t^3$ for $-\pi < t < \pi$.
(c) $f_4(t) = t^4$ for $-\pi < t < \pi$.

3. Given that

$$|t| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=\text{odd}} \frac{1}{n^2} \cos nt, \quad \text{for } -\pi < t < \pi$$

compute the Fourier series for the function

$$f(t) = t^2 \operatorname{sgn} t = \begin{cases} t^2 & \text{if } 0 < t < \pi \\ -t^2 & \text{if } -\pi < t < 0. \end{cases}$$

4. Compute the Fourier series for each of the following 2π-periodic functions. You should take advantage of Theorem 1 and the Fourier series already computed or given in Exercises 2 and 3.

(a) $g_1(t) = 2t - |t|, -\pi < t < \pi$.
(b) $g_2(t) = At^2 + Bt + C, -\pi < t < \pi$, where A, B, and C are constants.
(c) $g_3(t) = t(\pi - t)(\pi + t), -\pi < t < \pi$.

5. Let $f(t)$ be the 4-periodic function given by

$$f(t) = \begin{cases} -\frac{t}{2} & \text{if } -2 < t < 0 \\ \frac{t^2}{2} - \frac{t}{2} & \text{if } 0 \le t < 2. \end{cases}$$

The Fourier series of $f(t)$ is

$$f(t) \sim -\frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \left(\frac{(-1)^n}{n^2} \cos \frac{n\pi}{2} t + \frac{(-1)^n - 1}{\pi n^3} \sin \frac{n\pi}{2} t \right).$$

(a) Verify that the hypotheses of Theorem 3 are satisfied for $f(t)$.
(b) Compute the Fourier series of the derivative $f'(t)$.
(c) Verify that the hypotheses of Theorem 3 are not satisfied for $f'(t)$.

10.6 Applications of Fourier Series

The applications of Fourier series that will be considered will be applications to differential equations. In this section we will consider finding periodic solutions to constant coefficient linear differential equations with periodic forcing function. In the next chapter, Fourier series will be applied to solve certain partial differential equations.

Periodically Forced Differential Equations

Example 1. Find all 2π-periodic solutions of the differential equation

$$y' + 2y = f(t) \tag{1}$$

where $f(t)$ is a given piecewise smooth 2π-periodic function.

► Solution. Since $f(t)$ is piecewise smooth it can be expanded into a Fourier series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt). \tag{2}$$

If $y(t)$ is a 2π-periodic solution of (1), then it can also be expanded into a Fourier series

$$y(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos nt + B_n \sin nt).$$

Since $y(t)$ can be differentiated termwise,

$$y'(t) = \sum_{n=1}^{\infty} (-nA_n \sin nt + nB_n \cos nt).$$

Now substitute $y(t)$ into the differential equation (1) to get

$$\begin{aligned}
y' + 2y &= \sum_{n=1}^{\infty}(-nA_n \sin nt + nB_n \cos nt) \\
&\qquad + 2\left(\frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n \cos nt + B_n \sin nt)\right) \\
&= A_0 + \sum_{n=1}^{\infty}\left((2A_n + nB_n)\cos nt + (2B_n - nA_n)\sin nt\right) \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt).
\end{aligned}$$

Comparing coefficients of $\cos nt$ and $\sin nt$ shows that the coefficients A_n and B_n must satisfy the following equations:

$$A_0 = \frac{a_0}{2},$$

and for $n \geq 1$,

$$\begin{aligned}
2A_n + nB_n &= a_n \\
-nA_n + 2B_n &= b_n,
\end{aligned} \tag{3}$$

which can be solved to give

$$A_n = \frac{2a_n - nb_n}{n^2 + 4} \quad \text{and} \quad B_n = \frac{na_n + 2b_n}{n^2 + 4}. \tag{4}$$

Hence, the only 2π-periodic solution of (1) has the Fourier series expansion

$$y(t) = \frac{a_0}{4} + \sum_{n=1}^{\infty}\left(\frac{2a_n - nb_n}{n^2 + 4}\cos nt + \frac{na_n + 2b_n}{n^2 + 4}\sin nt\right).$$

For a concrete example of this situation, let $f(t)$ be the odd square wave of period 2π:

$$f(t) = \begin{cases} -1 & -\pi \leq t < 0, \\ 1 & 0 \leq t < \pi, \end{cases}; \qquad f(t + 2\pi) = f(t).$$

which has Fourier series

$$\begin{aligned}
f(t) &= \frac{4}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \frac{1}{7}\sin 7t + \cdots\right) \\
&= \frac{4}{\pi}\sum_{n=\text{odd}} \frac{\sin nt}{n}.
\end{aligned}$$

Since all of the cosine terms and the even sine terms are 0, (4) gives $A_n = B_n = 0$ for n even, and for n odd

$$A_n = \frac{-4}{\pi(n^2+4)} \quad \text{and} \quad B_n = \frac{8}{\pi n(n^2+4)}$$

so that the periodic solution is

$$y(t) = \frac{4}{\pi} \sum_{n=\text{odd}} \left(\frac{-1}{n^2+4} \cos nt + \frac{2}{n(n^2+4)} \sin nt \right).$$

Rewriting this in phase amplitude form gives

$$y(t) = \frac{4}{\pi} \sum_{n=\text{odd}} C_n \cos(nt - \delta_n)$$

where

$$C_n = \sqrt{A_n^2 + B_n^2} = \frac{4}{\pi(n^2+4)} \sqrt{1 + \frac{4}{n^2}}.$$

Thus, computing numerical values gives

$$\begin{aligned} C_1 &= 0.5985 \\ C_3 &= 0.1227 \\ C_5 &= 0.0493 \\ C_7 &= 0.0260 \\ C_9 &= 0.0160 \end{aligned}$$

The amplitudes are decreasing, but they are decreasing at a slow rate. This is typical of first order equations. ◀

Now consider a second order linear constant coefficient differential equation with $2L$-periodic forcing function

$$ay'' + by' + cy = f(t), \tag{5}$$

where $f(t)$ is a piecewise continuous $2L$-periodic with Fourier series

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi t}{L} + b_n \sin \frac{n\pi t}{L} \right). \tag{6}$$

To simplify notation, let $\omega = \pi/L$ denote the frequency. Then the Fourier series is written as

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n\omega t + b_n \sin n\omega t \right). \tag{7}$$

Assume that there is a $2L$-periodic solution to (5) that can be expressed in Fourier series form

$$y(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos n\omega t + B_n \sin n\omega t\right). \tag{8}$$

Since

$$\begin{aligned} y'(t) &= \sum_{n=1}^{\infty} \left(-n\omega A_n \sin n\omega t + n\omega B_n \cos n\omega t\right) \\ y''(t) &= \sum_{n=1}^{\infty} \left(-n^2\omega^2 A_n \cos n\omega t - n^2\omega^2 B_n \sin n\omega t\right) \end{aligned}$$

,

substituting into the differential equation (5) gives

$$\begin{aligned} &ay''(t) + by'(t) + cy(t) = \\ &\qquad c\frac{A_0}{2} + \sum_{n=1}^{\infty} \left[\left(\left(c - an^2\omega^2\right) A_n + bn\omega B_n\right) \cos n\omega t \right. \\ &\qquad\qquad \left. + \left(\left(c - an^2\omega^2\right) B_n - bn\omega A_n\right) \sin n\omega t\right] \\ &\qquad\qquad\qquad = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n\omega t + b_n \sin n\omega t\right). \end{aligned} \tag{9}$$

Equating coefficients of $\cos n\omega t$ and $\sin n\omega t$ leads to equations for A_n and B_n:

$$c\frac{A_0}{2} = \frac{a_0}{2}$$

and for $n \geq 1$,

$$\begin{aligned} \left(c - an^2\omega^2\right) A_n + bn\omega B_n &= a_n \\ -bn\omega A_n + \left(c - an^2\omega^2\right) B_n &= b_n. \end{aligned} \tag{10}$$

This system can be solved for A_n and B_n as long as the determinant of the coefficient matrix is not zero. That is, if

$$\begin{vmatrix} c - an^2\omega^2 & bn\omega \\ -bn\omega & c - an^2\omega^2 \end{vmatrix} = \left(c - an^2\omega^2\right)^2 + (bn\omega)^2 \neq 0.$$

If $b \neq 0$ this expression is always nonzero, while if $b = 0$ it is nonzero as long as $n\omega \neq \sqrt{c/a}$. These two conditions can be combined compactly into one using the characteristic polynomial $q(s) = as^2 + bs + c$ of the differential equation (5). Note that if $s = in\omega$ then

$$q(in\omega) = a(in\omega)^2 + bin\omega + c = -an^2\omega^2 + ibn\omega + c.$$

Thus, $q(in\omega) \neq 0$ if and only if $b \neq 0$ or $b = 0$ and $n\omega \neq \sqrt{c/a}$. In this case, solving (10) for A_n and B_n (using Cramer's rule) gives

$$\begin{aligned} A_n &= \frac{a_n\left(c - an^2\omega^2\right) - bn\omega b_n}{\left(c - an^2\omega^2\right)^2 + (bn\omega)^2} \\ B_n &= \frac{b_n\left(c - an^2\omega^2\right) + bn\omega a_n}{\left(c - an^2\omega^2\right)^2 + (bn\omega)^2} \end{aligned} \tag{11}$$

Thus we have arrived at the following result.

Theorem 2. *If $f(t)$ is a $2L$-periodic piecewise continuous function, the differential equation (5) with characteristic polynomial $q(s)$ has a unique $2L$-periodic solution whose Fourier series (8) has coefficients computed from those of $f(t)$ by (11), provided that $q(in\omega) \neq 0$ for all $n \geq 1$.*

Example 3. Find all 2-periodic solutions of

$$y'' + 10y = f(t) \tag{12}$$

where $f(t)$ is the 2-periodic odd square wave function

$$f(t) = \begin{cases} -1 & -1 \leq t < 0, \\ 1 & 0 \leq t < 1, \end{cases}; \qquad f(t+2) = f(t).$$

▶ Solution. The Fourier series for $f(t)$ is

$$f(t) \sim \frac{4}{\pi} \sum_{n=\text{odd}} \frac{\sin n\pi t}{n}.$$

Since $n\pi \neq 10$ for any $n \geq 1$, there is a unique 2-periodic solution $y_p(t)$ of (12) with Fourier series (whose coefficients are computed from (11))

$$y_p(t) = \frac{4}{\pi} \sum_{n=\text{odd}} \frac{1}{n(10 - n^2\pi^2)} \sin n\pi t.$$

If we calculate the first few terms of this Fourier series we see

$$\begin{aligned} y_p(t) &= \frac{4}{\pi}\left(\frac{1}{10 - \pi^2}\sin \pi t + \frac{1}{3(10 - 9\pi^2)}\sin 3\pi t + \frac{1}{5(10 - 25\pi^2)}\sin 5\pi t + \cdots\right) \\ &\cong 9.76443 \sin \pi t - 0.00538 \sin 3\pi t - 0.00108 \sin 5\pi t - 0.00038 \sin 7\pi t + \cdots \\ &\cong 9.76443 \sin \pi t + (\text{small terms}). \end{aligned}$$

Since all the terms in the Fourier series except for the first sine term are extremely small, the effect of the square wave input (forcing function) is an output that is essentially a single sinusoidal term

$$y_p(t) \cong 9.76443 \sin \pi t.$$

It is typical of second order differential equations to very rapidly reduce the effect of the higher order frequencies of the periodic input (forcing) function. ◀

Example 4. Suppose a mass-spring-dashpot system has mass $m = 1$ kg, spring constant $k = 25$ kg/sec^2, damping constant $\mu = 0.02$ kg/sec, and 2π-periodic forcing function $f(t)$ defined by

$$f(t) = \frac{\pi}{2} - |t|, \quad -\pi < t < \pi.$$

Thus, the equation of motion is

$$y'' + 0.02y' + 25y = f(t). \tag{13}$$

Find the unique 2π-periodic solution $y(t)$ for this problem.

▶ Solution. Start by expanding $f(t)$ into a Fourier series (see Example 4 of Section (10.2)):

$$\begin{aligned} f(t) &= \frac{4}{\pi}\left(\frac{\cos t}{1^2} + \frac{\cos 3t}{3^2} + \frac{\cos 5t}{5^2} + \frac{\cos 7t}{7^2} + \cdots\right) \\ &= \frac{4}{\pi}\sum_{n=\text{odd}} \frac{\cos nt}{n^2}. \end{aligned} \tag{14}$$

The unique 2π-periodic solution is

$$y(t) = \sum_{n=\text{odd}} A_n \cos nt + B_n \sin nt$$

where where coefficients A_n and B_n are computed from (11):

$$A_n = \frac{4(25-n^2)}{n^2\pi((25-n^2)^2 + (0.02n)^2)} \quad \text{and} \quad B_n = \frac{0.08}{n\pi((25-n^2)^2 + (0.02n)^2)}.$$

Using the phase amplitude formula, each term $y_n(t) = A_n \cos nt + B_n \sin nt$ can be represented in the form $C_n \sin(nt - \theta_n)$ where

$$C_n = \sqrt{A_n^2 + B_n^2} = \frac{4}{n^2\pi\sqrt{(25-n^2)^2 + (0.02n)^2}}$$

and the solution $y(t)$ can be expressed as

$$y(t) = \sum_{n=\text{odd}} C_n \sin(nt - \theta_n).$$

The first few numerical values of C_n are

$$\begin{aligned} C_1 &= 0.0530 \\ C_3 &= 0.0088 \\ C_5 &= 0.5100 \\ C_7 &= 0.0011 \\ C_9 &= 0.0003 \end{aligned}$$

Thus

$$\begin{aligned} y(t) \cong (0.0530)\sin(t+\theta_1) + (0.0088)\sin(3t+\theta_3) + (0.5100)\sin(5t+\theta_5) \\ + (0.0011)\sin(7t+\theta_7) + (0.0003)\sin(9t+\theta_9) + \cdots . \end{aligned}$$

The relatively large magnitude of C_5 is due to the fact that the quantity $(25-n^2)^2 + (0.02n)^2$ in the denominator is very small for $n = 5$, making $(0.5100)\sin(5t+\theta_5)$ the dominate term in the Fourier series expansion of $y(t)$. Thus, the dominant motion of a spring represented by the differential equation (13) almost a sinusoidal oscillation whose frequency is five times that of the periodic forcing function. ◀

Exercises

1. Let $f(t)$ be the 2-periodic function defined on $-1 < t < 1$ by

$$f(t) = \begin{cases} 0 & \text{if } -1 < t < 0 \\ 1 & \text{if } 0 < t < 1 \end{cases}.$$

Find all 2-periodic solutions to $y'' + 4y = f(t)$.

2. Find all periodic solutions of

$$y'' + y' + y = \sum_{n=1}^{\infty} \frac{1}{n^3} \cos nt.$$

3. Find the general solution of

$$y'' + y = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nt$$

Hint: First solve $y'' + y = \cos t$ by undetermined coefficients. Then let $f(t) = \sum_{n=2}^{\infty} n^{-2} \cos nt$ and solve $y'' + y = f(t)$.

4–7. Find the unique periodic solution of the given differential equation.

4. $y'' + 3y = f(t)$ where $f(t)$ is the 2π-periodic function defined by $f(t) = 5$ if $0 < t < \pi$ and $f(t) = -5$ if $\pi < t < 0$.

5. $y'' + 10y = f(t)$ where $f(t)$ is the 4-periodic even function defined by $f(t) = 5$ if $0 < t < 1$ and $f(t) = 0$ if $1 < t < 2$.

6. $y'' + 5y = f(t)$ where $f(t)$ is the 2-periodic odd function defined by $f(t) = t$ if $0 < t < 1$.

7. $y'' + 5y = f(t)$ where $f(t)$ is the 2-periodic even function defined by $f(t) = t$ if $0 < t < 1$.

8–10. The values of m, μ, and k for a mass-spring-dashpot system are given. Find the $2L$-periodic motion expressed in phase amplitude form

$$y_p(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} C_n \sin(\frac{n\pi t}{L} - \theta_n)$$

of the mass under the $2L$-periodic forcing function $f(t)$. Compute the first three nonzero terms C_n.

8. $m = 1$, $\mu = 0.1$, $k = 4$; $f(t)$ is the force in Problem 4.

9. $m = 2$, $\mu = 0.1$, $k = 18$; $f(t)$ is the 2π-periodic odd function defined by $f(t) = 2t$ if $0 < t < \pi$.

10. $m = 1$, $\mu = 1$, $k = 10$; $f(t)$ is the force in Problem 5.