CHAPTER 10

Fourier Series

10.1 INTRODUCTION

In connection with the solution of the heat equation in Section 6.2.1, we still have to show how to choose constants b_n for $n = 1, 2, 3, \ldots$ in such a way that a given function f can be expressed as a trigonometric series of the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}. \tag{1}$$

This, and the more general problem of expressing a given function f as a series of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}\right), \tag{2}$$

will be the subject matter of this chapter.

Series like the ones which appear in the right-hand sides of (1) and (2) are called *trigonometric series* or *Fourier series* in honor of the French scientist J. B. Fourier. Fourier discovered an ingenious method for computing the coefficients a_n and b_n of (2) and made systematic use of such series in connection with his work on heat conduction in 1807 and 1811. However, Fourier's work lacked mathematical rigor, thus leading him to the false conclusion that any arbitrary function can be expressed as a series of the form (2). This is not true, as we will see in Section 10.4. Consequently, when Fourier published his results, they were considered to be nonsense by many of his contemporaries. Since then, the theory of Fourier series has been developed on a rigorous basis and has become an indispensable tool in many areas of scientific work.

10.2 PERIODICITY AND ORTHOGONALITY OF SINES AND COSINES

In this section we study the periodic character and orthogonality properties of the functions

$$\cos \frac{n\pi x}{l} \quad \text{and} \quad \sin \frac{n\pi x}{l} \quad \text{for} \quad n = 1, 2, 3, \ldots; l > 0, \tag{1}$$

which are the building blocks of Fourier series.

Let us recall that a function f is called *periodic with period $T > 0$* if for all x in the domain of the function

$$f(x+T) = f(x). \tag{2}$$

Geometrically, this means that the graph of f repeats itself in successive intervals of length T. The functions $\sin x$ and $\cos x$ are simple examples of periodic functions with period 2π. A constant function is also periodic with period any positive number. Other examples of periodic functions are the functions in (1) with period $2l$ (see Remark 1) and the functions shown graphically in Figure 10.1. The function in Figure 10.1(a) has period 2 and that in Figure 10.1(b) has period 1.

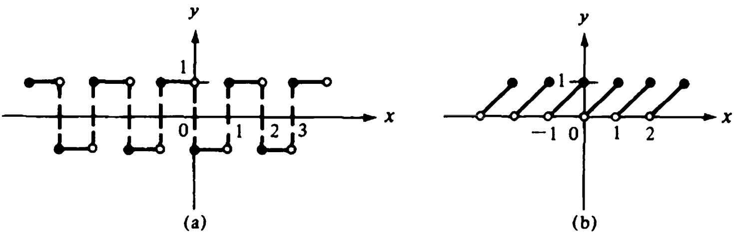


Figure 10.1

Periodic functions appear in a variety of real life situations. Alternating currents, the vibrations of a spring, sound waves, and the motion of a pendulum are examples of periodic functions.

Periodic functions have many periods. For example, 2π, 4π, 6π, . . . are periods of $\sin x$ and $\cos x$. More generally, it follows from (2) that if f has period T, then

$$f(x) = f(x+T) = f(x+2T) = f(x+3T) = \dots,$$

which means that $2T, 3T, \dots$ are also periods of f. The smallest positive number T for which Eq. (2) holds, if it exists, is called the *fundamental period* of f. For example, the fundamental period of $\sin x$ and $\cos x$ is 2π, and in general (see Theorem 1) the fundamental period of each function in (1) is $2l/n$. On the other hand, a constant function has no fundamental period. This is because any positive number, no matter how small, is a period.

THEOREM 1

The functions $\cos px$ and $\sin px$, $p > 0$, are periodic with fundamental period $2\pi/p$. In particular, the functions

$$\cos\frac{n\pi x}{l} \quad \text{and} \quad \sin\frac{n\pi x}{l}, \qquad n = 1, 2, 3, \dots,$$

where l is a positive number, are periodic with fundamental period $T = 2l/n$.

Proof We give the proof for the function $\cos px$. The proof for $\sin px$ is similar. Assume that T is a period of $f(x) = \cos px$. Then the statement $f(x+T) = f(x)$ for all x is equivalent to $\cos(px+pT) = \cos px$ or, after expanding the cosine of the sum $px + pT$,

$$\cos px \cos pT - \sin px \sin pT = \cos px, \text{ for all } x. \tag{3}$$

But (3) is true for all x if and only if $\cos pT = 1$ and $\sin pT = 0$. That is, $pT = 2n\pi$, with $n = 1, 2, 3, \ldots$ (recall that p and T are positive). Thus $T = 2n\pi/p$, $n = 1, 2, \ldots$, and so the fundamental period (the least positive) of $\cos px$ is $2\pi/p$. Clearly, each of the functions listed in (1) is periodic with fundamental period $2\pi/(n\pi/l) = 2l/n$.

REMARK 1 Since the functions in (1) have (fundamental) period $2l/n$, it follows that each one has also period $2l$.

Next we turn to the question of orthogonality of the functions in (1). Recall the definition of orthogonal functions given in Section 6.2. Two functions f and g defined and continuous in an interval $a \le x \le b$ are said to be *orthogonal* on $a \le x \le b$ if

$$\int_a^b f(x)g(x)dx = 0.$$

The following result establishes the fact that the functions listed in (1) are *mutually orthogonal* in the interval $-l \le x \le l$. This means that any two distinct functions from (1) are orthogonal in $-l \le x \le l$. More precisely, we have Theorem 2.

THEOREM 2

The functions

$$\cos\frac{n\pi x}{l} \quad \text{and} \quad \sin\frac{n\pi x}{l} \quad \text{for} \quad n = 1, 2, 3, \ldots; l > 0$$

satisfy the following orthogonality properties in the interval $-l \le x \le l$.

$$\int_{-l}^{l} \cos\frac{n\pi x}{l}\cos\frac{k\pi x}{l}dx = \begin{cases} 0 & \text{if } n \ne k \\ l & \text{if } n = k. \end{cases} \tag{4}$$

$$\int_{-l}^{l} \sin\frac{n\pi x}{l}\sin\frac{k\pi x}{l}dx = \begin{cases} 0 & \text{if } n \ne k \\ l & \text{if } n = k. \end{cases} \tag{5}$$

$$\int_{-l}^{l} \cos\frac{n\pi x}{l}\sin\frac{k\pi x}{l}dx = 0 \qquad \text{for all } n, k. \tag{6}$$

Proof We can (and do) immediately establish (6) by using the fact that the integrand is an odd function and so its integral over the interval $-l \le x \le l$ (which is symmetric with respect to the origin) is zero. (For a review of odd and even functions, see Section 10.5.)

Next we prove (4). For $n = k$ we have, using the identity $\cos^2 x = \dfrac{1 + \cos 2x}{2}$,

$$\int_{-l}^{l} \cos\frac{n\pi x}{l}\cos\frac{k\pi x}{l}\,dx = \int_{-l}^{l}\left(\cos\frac{n\pi x}{l}\right)^2 dx$$

$$= \int_{-l}^{l} \tfrac{1}{2}\left[1 + \cos\frac{2n\pi x}{l}\right]dx = \tfrac{1}{2}\left[x + \frac{l}{2n\pi}\sin\frac{2n\pi x}{l}\right]\Bigg|_{-l}^{l} = l.$$

Now for $n \neq k$ we find, using the identity

$$\cos x \cdot \cos y = \tfrac{1}{2}[\cos(x+y) + \cos(x-y)],$$

$$\int_{-l}^{l} \cos\frac{n\pi x}{l}\cos\frac{k\pi x}{l}\,dx = \int_{-l}^{l}\tfrac{1}{2}\left[\cos\frac{(n+k)\pi x}{l} + \cos\frac{(n-k)\pi x}{l}\right]dx$$

$$= \left[\tfrac{1}{2}\frac{l}{(n+k)\pi}\sin\frac{(n+k)\pi x}{l} + \tfrac{1}{2}\frac{l}{(n-k)\pi}\sin\frac{(n-k)\pi x}{l}\right]\Bigg|_{-l}^{l} = 0.$$

The proof of (5) is similar to that of (4). In the case $n = k$ we need the identity $\sin^2 x = (1 - \cos 2x)/2$, and in the case $n \neq k$ we need the identity

$$\sin x \sin y = \tfrac{1}{2}[\cos(x-y) - \cos(x+y)].$$

EXERCISES

In Exercises 1 through 12, answer true or false.

1. The functions $3\sin\dfrac{x}{2}$ and $2\cos\dfrac{x}{2}$ have fundamental period equal to 4π.
2. The function $2 + 3\sin x + 4\cos x$ has period 3π.
3. The function $x\cos x$ is periodic.
4. The function $\sin^2 x \cos x$ has period 2π.
5. The functions $\sin x$ and x^2 are orthogonal in the interval $-1 \le x \le 1$.
6. The functions $\cos\dfrac{n\pi x}{l}$, $n = 1, 2, 3, \ldots$ are mutually orthogonal in the interval $0 \le x \le l$.
7. $\displaystyle\int_{-l}^{l}\sin\frac{n\pi x}{l}\,dx = \int_{0}^{2l}\sin\frac{n\pi x}{l}\,dx = 2\int_{0}^{l}\sin\frac{n\pi x}{l}\,dx.$
8. $\displaystyle\int_{-l}^{l}\cos\frac{n\pi x}{l}\,dx = \int_{0}^{2l}\cos\frac{n\pi x}{l}\,dx = 2\int_{0}^{l}\cos\frac{n\pi x}{l}\,dx.$

9. If the series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}\right)$$

converges, the sum would be a periodic function with period $2l/n$.

10. Any solution of the differential equation $y''' + y' = 0$ has fundamental period 2π.

11. Any solution of the differential equation $y''' + y' = 0$ has period 2π.

12. The differential equation $y'' - y = \sin x$ has periodic solutions.

In Exercises 13 through 16, graph the given function.

13. $f(x) = \begin{cases} 1, & -1 \leq x < 0 \\ -1, & 0 \leq x < 1 \end{cases}; \quad f(x+2) = f(x)$

14. $f(x) = x, \quad -\pi \leq x < \pi; \quad f(x+2\pi) = f(x)$

15. $f(x) = x, \quad 0 < x \leq 1; \quad f(x+1) = f(x)$

16. $f(x) = \begin{cases} -3, & -2 \leq x < -1 \\ 0, & -1 \leq x \leq 1 \\ 3, & 1 < x < 2 \end{cases}; \quad f(x+4) = f(x)$

17. Assume that the functions f and g are defined for all x and that they are periodic with common period T. Show that for any constants a and b, the functions $af + bg$ and fg are also periodic with period T.

18. Assume that the function f is defined for all x and is periodic with period T. Show that if f is integrable in the interval $0 \leq x \leq T$, then for any constant c,

$$\int_0^T f(x)\,dx = \int_c^{c+T} f(x)\,dx.$$

19. Find a necessary and sufficient condition for all solutions of the differential equation $y'' + py = 0$, with p constant to be periodic. What is the period?

20. Find a necessary and sufficient condition for all solutions of the system with constant coefficients

$$\begin{aligned} \dot{x} &= ax + by \\ \dot{y} &= cx + dy \end{aligned}$$

to be periodic.

10.3 FOURIER SERIES

Let us begin by assuming that a given function f, defined in the interval

$-l \le x \le l$ and outside of this interval by $f(x + 2l) = f(x)$, so that f has period $2l$, can be expressed as a trigonometric series of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right). \tag{1}$$

We want to compute the coefficients a_n, $n = 0, 1, 2, \ldots$, and b_n, $n = 1, 2, 3, \ldots$, of (1). Consider the orthogonality properties of the functions $\cos(n\pi x/l)$ and $\sin(n\pi x/l)$ for $n = 1, 2, 3, \ldots$ in the interval $-l \le x \le l$.

(i) To compute the coefficients a_n for $n = 1, 2, 3, \ldots$, multiply both sides of (1) by $\cos(k\pi x/l)$, with k a positive integer, then integrate from $-l$ to l. For the moment we assume that the integrals exist and that it is legal to integrate the series term by term. Then, using (4) and (6) from Section 10.2, we find

$$\int_{-l}^{l} f(x) \cos \frac{k\pi x}{l}\, dx = \frac{a_0}{2} \underbrace{\int_{-l}^{l} \cos \frac{k\pi x}{l}\, dx}_{0}$$

$$+ \sum_{n=1}^{\infty} \Bigg[a_n \underbrace{\int_{-l}^{l} \cos \frac{n\pi x}{l} \cos \frac{k\pi x}{l}\, dx}_{\substack{0 \text{ if } n \neq k \\ l \text{ if } n = k}} + b_n \underbrace{\int_{-l}^{l} \sin \frac{n\pi x}{l} \cos \frac{k\pi x}{l}\, dx}_{0} \Bigg] = a_k l.$$

Thus,

$$a_k = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{k\pi x}{l}\, dx, \qquad k = 1, 2, 3, \ldots$$

or, replacing k by n,

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l}\, dx, \qquad n = 1, 2, 3, \ldots. \tag{2}$$

(ii) To compute a_0, integrate both sides of (1) from $-l$ to l. Then

$$\int_{-l}^{l} f(x)dx = \frac{a_0}{2} \underbrace{\int_{-l}^{l} dx}_{2l} + \sum_{n=1}^{\infty} \Bigg[a_n \underbrace{\int_{-l}^{l} \cos \frac{n\pi x}{l}\, dx}_{0} + b_n \underbrace{\int_{-l}^{l} \sin \frac{n\pi x}{l}\, dx}_{0} \Bigg]$$

$$= a_0 l.$$

Hence,

$$a_0 = \frac{1}{l} \int_{-l}^{l} f(x)dx. \tag{3}$$

That is, a_0 is twice the average value of the function f over the interval $-l \le x \le l$. Note that the value of a_0 can be obtained from formula (2) for $n = 0$. Of course, if the constant a_0 in (1) were not divided by 2, we would need a separate formula for a_0. As it is, all a_n are given by a single formula, namely,

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l}\, dx, \qquad n = 0, 1, 2, \ldots. \tag{4}$$

(iii) Finally, to compute b_n for $n = 1, 2, 3, \ldots$, multiply both sides of (1) by $\sin(k\pi x/l)$, with k a positive integer; then integrate from $-l$ to l. Using (6) and (5) from Section 10.2, we find

$$\int_{-l}^{l} f(x)\sin\frac{k\pi x}{l}\,dx = \frac{a_0}{2}\underbrace{\int_{-l}^{l}\sin\frac{k\pi x}{l}\,dx}_{0}$$

$$+\sum_{n=1}^{\infty}\left[a_n\underbrace{\int_{-l}^{l}\cos\frac{n\pi x}{l}\sin\frac{k\pi x}{l}\,dx}_{0} + b_n\underbrace{\int_{-l}^{l}\sin\frac{n\pi x}{l}\sin\frac{k\pi x}{l}\,dx}_{\substack{0 \text{ if } n\neq k\\ l \text{ if } n=k}}\right] = b_k l.$$

Thus, replacing k by n, we find

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx, \qquad n = 1, 2, 3, \ldots. \tag{5}$$

When the coefficients a_n and b_n of (1) are given by the formulas (4) and (5) above, then the right-hand side of (1) is called the *Fourier series of the function* f over the interval of definition of the function. The formulas (4) and (5) are known as the *Euler-Fourier formulas*, and the numbers a_n and b_n are called the *Fourier coefficients* of f. We will write

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right) \tag{6}$$

to indicate that the right-hand side of (6) is the Fourier series of the function f.

Before we present any examples, the following remarks are in order.

REMARK 1 So far we proved that, if the right-hand side of (1) converges and has sum $f(x)$, if f is integrable in the interval $-l \le x \le l$, and if the term by term integrations could be justified, then the coefficients a_n and b_n in (1) must be given by the formulas (4) and (5) respectively. On the other hand, if a function f is given and if we formally write down its Fourier series, there is no guarantee that the series converges. Even if the Fourier series converges, there is no guarantee that its sum is equal to $f(x)$. The convergence of the Fourier series and how its sum is related to $f(x)$ will be investigated in the next section.

REMARK 2 To compute the Fourier coefficients a_n and b_n, we *only* need the values of f in the interval $-l \le x \le l$ and the assumption that f is integrable there. It is a fact, however, that an integral is not affected by changing the values of the integrand at a finite number of points. In particular we can compute the Fourier coefficients a_n and b_n if f is integrable in $-l \le x \le l$, although the function may not be defined, or may be discontinuous at a finite number of points in that interval. Of course the interval does not have to be closed; it may be open or closed at one end and open at the other.

REMARK 3 When the series in (1) converges for all x, its sum must be a periodic function of period $2l$. This is because every term of the series is periodic with period $2l$. For this reason Fourier series is an indispensable tool for the study of periodic phenomena. Assume that a function f is not periodic and is only defined in the interval $-l \leq x < l$ (or $-l < x \leq l$, or $-l < x < l$). We can write its Fourier series in $-l \leq x < l$. We also have the choice of extending f outside of this interval as a periodic function with period $2l$. The periodic extension of f, F, agrees with f in the interval $-l \leq x < l$. Therefore, a function f, defined in $-l \leq x < l$, and its periodic extension, which is defined for all x, have identical Fourier series. (See Example 4.) Finally we should mention that if f is defined in a closed interval $-l \leq x \leq l$ and if $f(-l) \neq f(l)$, then f cannot be extended periodically. In such a case we can either ignore (as we do in this book) or modify the values of f at $\pm l$ and proceed with the periodic extension.

REMARK 4 When f is periodic with period $2l$, the Fourier coefficients of f can be determined from formulas (4) and (5) or, equivalently, from

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx, \quad n = 0, 1, 2, \ldots \tag{4'}$$

and

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx, \quad n = 1, 2, 3, \ldots, \tag{5'}$$

where c is any real number. This follows immediately from Exercise 18 of Section 10.2, with $T = 2l$ and $c = -l$. Observe that for $c = -l$, formulas (4′) and (5′) reduce to (4) and (5) respectively.

EXAMPLE 1 Compute the Fourier series of the function

$$f(x) = \begin{cases} 0, & -\pi \leq x < \dfrac{\pi}{2} \\ 1, & \dfrac{\pi}{2} \leq x < \pi \end{cases}; \qquad f(x + 2\pi) = f(x).$$

Solution Since the period of f is 2π, we have $2l = 2\pi$, $l = \pi$. Hence, the Fourier series of f is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

with

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{\pi/2}^{\pi} \cos nx\,dx, \qquad n = 0, 1, 2, \ldots \tag{7}$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{\pi/2}^{\pi} \sin nx\,dx, \qquad n = 1, 2, 3, \ldots. \tag{8}$$

To evaluate the integral in (7), we have to distinguish between the two cases $n = 0$ and $n \neq 0$. For $n = 0$, the integrand is $\cos 0 = 1$, and so

$$a_0 = \frac{1}{\pi}\int_{\pi/2}^{\pi} dx = \frac{1}{\pi}x\Big|_{\pi/2}^{\pi} = \frac{1}{\pi}\left(\pi - \frac{\pi}{2}\right) = \frac{1}{\pi}\frac{\pi}{2} = \frac{1}{2}.$$

On the other hand, for $n \neq 0$, and so for $n = 1, 2, 3, \ldots$, we find

$$a_n = \frac{1}{\pi}\frac{\sin nx}{n}\Big|_{\pi/2}^{\pi} = \frac{1}{n\pi}\left(\sin n\pi - \sin\frac{n\pi}{2}\right) = -\frac{\sin(n\pi/2)}{n\pi}, \qquad n = 1, 2, 3, \ldots.$$

From (8) we find, for $n = 1, 2, 3, \ldots,$

$$b_n = \frac{1}{\pi}\left(-\frac{\cos nx}{n}\right)\Big|_{\pi/2}^{\pi} = -\frac{1}{n\pi}[\cos n\pi - \cos(n\pi/2)]$$

$$= \frac{\cos(n\pi/2) - (-1)^n}{n\pi}, \qquad n = 1, 2, 3, \ldots.$$

Hence, the Fourier series of f is

$$f(x) \sim \frac{1}{4} + \sum_{n=1}^{\infty}\left[-\frac{\sin(n\pi/2)}{n\pi}\cos nx + \frac{\cos(n\pi/2) - (-1)^n}{n\pi}\sin nx\right]$$

$$= \frac{1}{4} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\{\sin(n\pi/2)\cos nx + [(-1)^n - \cos(n\pi/2)]\sin nx\}.$$

EXAMPLE 2 Find the Fourier series of the function

$$f(x) = x^2, \qquad -1 < x \leq 1; \qquad f(x + 2) = f(x).$$

Solution Since the period of f is 2, we have $2l = 2$, $l = 1$. Hence the Fourier series of f is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos n\pi x + b_n\sin n\pi x),$$

with

$$a_n = \int_{-1}^{1} x^2\cos n\pi x\,dx, \qquad n = 0, 1, 2, \ldots. \tag{9}$$

and

$$b_n = \int_{-1}^{1} x^2\sin n\pi x\,dx, \qquad n = 1, 2, 3, \ldots. \tag{10}$$

From (9) we find, for $n = 0$,

$$a_0 = \int_{-1}^{1} x^2 dx = \frac{x^3}{3}\bigg|_{-1}^{1} = \frac{2}{3}.$$

On the other hand, for $n = 1, 2, 3, \ldots$, and integrating by parts twice, or using the integral tables in the book, we find

$$\begin{aligned} a_n &= \left(\frac{x^2}{n\pi}\sin n\pi x + \frac{2x}{n^2\pi^2}\cos n\pi x - \frac{2}{n^3\pi^3}\sin n\pi x\right)\bigg|_{-1}^{1} \\ &= \frac{4\cos n\pi}{n^2\pi^2} = \frac{4(-1)^n}{n^2\pi^2}, \qquad n = 1, 2, 3, \ldots. \end{aligned}$$

Again from the tables in the book, or integrating by parts, or using the fact that the integrand in (10) in an odd function of x, we find

$$b_n = 0, \qquad n = 1, 2, 3, \ldots.$$

Hence, the Fourier series of f is

$$f(x) \sim \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2\pi^2}\cos n\pi x = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x.$$

EXAMPLE 3 Determine the Fourier series of the function

$$f(x) = x, \qquad -\pi < x \le \pi.$$

Solution Here f is only defined in the interval $-\pi < x \le \pi$. Hence $2l = 2\pi$ or $l = \pi$. Its Fourier series in this interval is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

with

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos nx \, dx, \qquad n = 0, 1, 2, \ldots \tag{11}$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx, \qquad n = 1, 2, 3, \ldots. \tag{12}$$

Since the integrand in (11) is an odd function of x for all n, we have

$$a_n = 0, \qquad n = 0, 1, 2, \ldots.$$

Integrating by parts or using the tables in the book, we find

$$b_n = \frac{1}{\pi}\left(\frac{1}{n^2}\sin nx - \frac{x}{n}\cos nx\right)\bigg|_{-\pi}^{\pi} = -\frac{2}{n}(-1)^n, \qquad n = 1, 2, 3, \ldots.$$

Hence, the Fourier series of $f(x) = x$ in the interval $-\pi < x \leq \pi$ is

$$x \sim \sum_{n=1}^{\infty} -\frac{2}{n}(-1)^n \sin nx = 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right).$$

EXAMPLE 4 Find the Fourier series of each of the following functions:

(i)	$f(x) = x,$	$-\pi \leq x < \pi$	
(ii)	$f(x) = x,$	$-\pi < x < \pi$	
(iii)	$f(x) = x,$	$-\pi < x \leq \pi;$	$f(x + 2\pi) = f(x)$
(iv)	$f(x) = x,$	$-\pi \leq x < \pi;$	$f(x + 2\pi) = f(x)$
(v)	$f(x) = x,$	$-\pi \leq x \leq \pi;$	$f(x + 2\pi) = f(x).$

Solution As we explained in Remark 3, or as we can see directly from formulas (11) and (12), all the above functions and the function of Example 3 have identical Fourier series, namely,

$$f(x) \sim \sum_{n=1}^{\infty} -\frac{2}{n}(-1)^n \sin nx.$$

EXERCISES

In Exercises 1 through 20, find the Fourier series of the given function.

1. $f(x) = x,\ -1 \leq x \leq 1; f(x + 2) = f(x)$

2. $f(x) = |x|,\ -\pi \leq x < \pi; f(x + 2\pi) = f(x)$

3. $f(x) = \begin{cases} 0, & -\pi \leq x < 0 \\ 1, & 0 \leq x < \pi \end{cases}; f(x + 2\pi) = f(x)$

4. $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$

5. $f(x) = x,\ 0 < x \leq 2\pi; f(x + 2\pi) = f(x)$
[*Hint:* Use formulas (4′) and (5′) with $c = 0$.]

6. $f(x) = x^2,\ -\pi \leq x \leq \pi$

7. $f(x) = x^2, 0 \leq x < 2\pi; f(x + 2\pi) = f(x)$

8. $f(x) = x^2,\ -l < x \leq l; f(x + 2l) = f(x)$

9. $f(x) = 2\cos^2 x,\ -\pi \leq x \leq \pi; f(x + 2\pi) = f(x)$

10. $f(x) = 2\sin^2 x,\ -\pi < x \leq \pi$

11. $f(x) = \sin 2x,\ -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

12. $f(x) = \cos 2x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$

13. $f(x) = x^3,\ -\pi < x \le \pi;\ f(x + 2\pi) = f(x)$

14. $f(x) = x^3,\ -1 \le x \le 1$

15. $f(x) = \cos\dfrac{x}{2},\ -\pi \le x < \pi$

16. $f(x) = \cos px,\ -\pi < x < \pi\ (p \ne \text{integer})$

17. $f(x) = \begin{cases} 0, & -2 < x \le -1 \\ 1, & -1 < x \le 1 \\ 0, & 1 < x \le 2 \end{cases};\qquad f(x+4) = f(x)$

18. $f(x) = \begin{cases} -1, & -2 \le x < -1 \\ 1, & -1 \le x < 1 \\ -1, & 1 \le x < 2 \end{cases};\qquad f(x+4) = f(x)$

19. $f(x) = \begin{cases} 0, & -l < x < 0 \\ 1, & 0 \le x < l \end{cases};\qquad f(x+2l) = f(x)$

20. $f(x) = e^x,\ -1 \le x < 1;\qquad f(x + 2) = f(x)$

10.4 CONVERGENCE OF FOURIER SERIES

Assume that a function f is defined in the interval $-l \le x < l$ and outside this interval by $f(x + 2l) = f(x)$, so that f has period $2l$. In Section 10.3 we defined the Fourier series of f,

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right), \tag{1}$$

where the Fourier coefficients a_n and b_n of f are given by the Euler-Fourier formulas

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx,\ n = 0, 1, 2, \ldots \tag{2}$$

and

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx,\ n = 1, 2, 3, \ldots. \tag{3}$$

When Fourier announced his famous theorem to the Paris Academy in 1807, he claimed that *any* function f could be represented by a series of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right), \tag{4}$$

where the coefficients a_n and b_n are given by (2) and (3). Fourier was wrong in asserting that (4) is true without any restrictions on the function f. As we will see in Theorem 1, there is a huge class of functions for which (4) fails at the points of discontinuities of the functions. Examples are also known of functions whose Fourier series diverge at "almost" every point. Sufficient conditions for (4) to be true were given by Dirichlet in 1829. However, necessary and sufficient conditions for (4) to hold have not been discovered.

In this section we state conditions which are sufficient to insure that the Fourier series converges for all x and furthermore that the sum of the series is equal to the value $f(x)$ at each point where f is continuous. These conditions, although not the most general sufficient conditions known today, are, nevertheless, generally satisfied in practice.

DEFINITION 1

A function f is said to be piecewise continuous on an interval I if I can be subdivided into a finite number of subintervals, in each of which f is continuous and has finite left- and right-hand limits. An example of a piecewise continuous function is shown graphically in Figure 10.2. Clearly, a piecewise continuous function on an interval I has a finite number of discontinuities on I. Such discontinuities (where the left- and right-hand limits exist but are unequal) are called jump discontinuities. The notation $f(c-)$ denotes the limit of $f(x)$ as $x \to c$ from the left. That is,

$$f(c-) = \lim_{h\to 0-} f(x+h).$$

Similarly we write $f(c+)$ to denote the limit of $f(x)$ as $x \to c$ from the right. If f is continuous at c, then

$$f(c-) = f(c+) = f(c). \tag{5}$$

THEOREM 1

Assume that f is a periodic function with period $2l$ and such that f and f' are piecewise continuous on the interval $-l \le x \le l$. Then the Fourier series of f converges to the value $f(x)$ at each point x where f is continuous, and to the average $[f(x-) + f(x+)]/2$ of the left- and right-hand limits at each point x where f is discontinuous.

The hypotheses of the above theorem[1] are known by the name *Dirichlet conditions*. Hence, if f satisfies the Dirichlet conditions, then

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right) = \begin{cases} f(x) \text{ if } x \text{ is point of continuity of } f \\ \dfrac{f(x-)+f(x+)}{2} \text{ if } x \text{ is point of discontinuity} \end{cases} \tag{6}$$

[1]For a proof of Theorem 1 see, for example, W. Kaplan, *Advanced Calculus* (Reading, Mass.: Addison-Wesley Publishing Co., 1973).

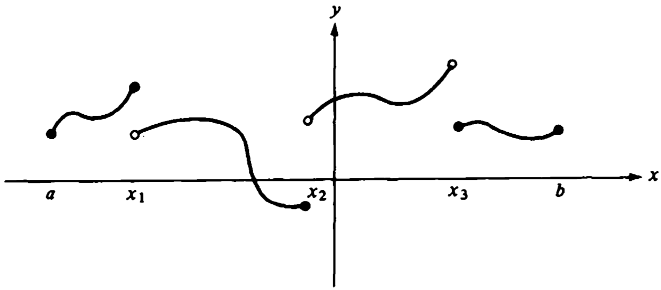


Figure 10.2

where a_n and b_n are given by (2) and (3). It follows from (6) that (4) is, in general, false at the point where f is discontinuous. On the other hand, if f is continuous everywhere and satisfies the Dirichlet conditions, then (4) is true for all x. Unless (4) is true for all x, we will continue using the symbol $\sim$ to indicate that the right-hand side is the Fourier series of the function to the left.

Using (5) we can write (6) in the form

$$\frac{f(x-) + f(x+)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right) \tag{6'}$$

which is true for all x. In fact, if x is a point of discontinuity of f, (6′) agrees with (6); if x is a point of continuity, we have $f(x-) = f(x+) = f(x)$, and the left-hand side of (6′) reduces to $f(x)$.

The Remarks 2, 3, and 4 of the last section are relevant to this section as well.

The conclusion of Theorem 1 is also true for functions f which are only defined on an interval I with endpoints $-l$ and $+l$, provided that f and f' are piecewise continuous on I. Then the periodic extension of f, which agrees with f on I, satisfies the Dirichlet conditions. Furthermore, the Fourier series of f and its periodic extension are identical. The periodic extension of f can also be utilized in finding the sum of the Fourier series of f at the endpoints $\pm l$. In fact, if F denotes the periodic extension of f, then, from (6′), the sum of the Fourier series of f at l is

$$\frac{F(l-) + F(l+)}{2} = \frac{f(l-) + f(-l+)}{2}$$

and at $-l$ is

$$\frac{F(-l-) + F(-l+)}{2} = \frac{f(l-) + f(-l+)}{2}.$$

Thus, the sum of the Fourier series of f at each of the endpoints $\pm l$ is $\frac{1}{2}[f(l-) + f(-l+)]$.

EXAMPLE 1 Find the Fourier series of the function

$$f(x) = \begin{cases} -1, & -\pi < x < \pi \\ 1, & 0 < x < \pi \end{cases}; \qquad f(x + 2\pi) = f(x).$$

Sketch for a few periods the graph of the function to which the series converges.

Solution The function f, whose graph is known as a *square wave* of period 2π and amplitude 1, satisfies the Dirichlet conditions (the hypotheses of Theorem 1) with $l = \pi$. In fact, the only points in the interval $-\pi \leq x \leq \pi$ where f or f' is not continuous are $x = 0$, and $x = \pm\pi$; the left- and right-hand limits at these points exist and are finite. [*Note:* $f'(x) = 0$ in $-\pi < x < 0$ and $0 < x < \pi$.] The graph of f is sketched in Figure 10.3.

The function f is continuous everywhere except at the points $0, \pm\pi, \pm 2\pi, \ldots$, where f is not even defined. From Theorem 1, the Fourier series of f converges to $f(x)$ at each point except $0, \pm\pi, \pm 2\pi, \ldots$. At each of the points $0, \pm\pi, \pm 2\pi, \ldots$, the Fourier series converges to the average of the left- and right-hand limit, which in this case is 0. Therefore, the graph of the function to which the Fourier series of f converges is now completely known. It is identical to f everywhere except at the discontinuities of f, where the value of the Fourier series is zero. See Figure 10.4.

Next we compute the Fourier series of f. Here $l = \pi$ and

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

with

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = -\frac{1}{\pi}\int_{-\pi}^{0} \cos nx\, dx + \frac{1}{\pi}\int_{0}^{\pi} \cos nx\, dx = 0,$$
$$n = 0, 1, 2, \ldots$$

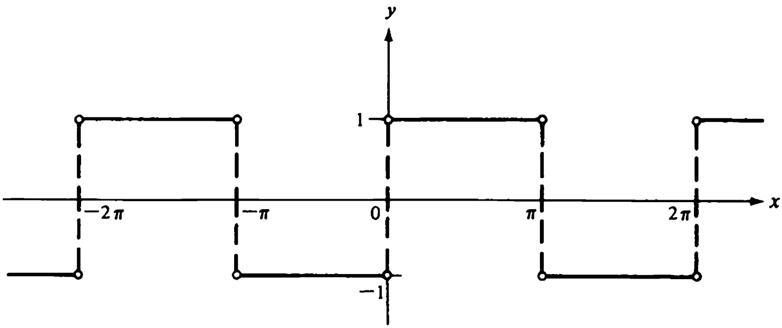


Figure 10.3

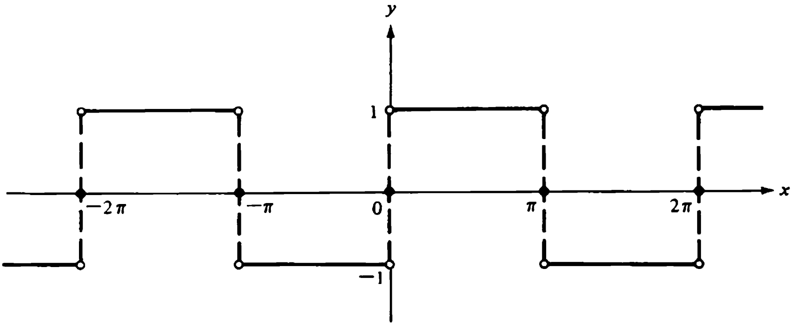


Figure 10.4

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = -\frac{1}{\pi}\int_{-\pi}^{0}\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}\sin nx\,dx$$

$$= \frac{2[1-(-1)^n]}{n\pi}, \qquad n = 1, 2, 3, \ldots .$$

Hence,

$$f(x) \sim \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right).$$

The symbol $\sim$ can be replaced by the equality sign everywhere except for $x = 0, \pm\pi, \pm 2\pi, \ldots$. Applying this idea to Fourier series often leads to interesting results. For example, in the above series, f is continuous and equal to 1 for all x in the interval $0 < x < \pi$. This leads to the trigonometric identity

$$\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots = \frac{\pi}{4}, \qquad 0 < x < \pi,$$

from which, for $x = \pi/2$, we find

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots .$$

EXAMPLE 2 Find the Fourier series of the function

$$f(x) = x, \qquad 0 \le x < 2\pi; \qquad f(x+2\pi) = f(x).$$

Sketch for a few periods the graph of the function to which the series converges.

Solution The graph of f is shown in Figure 10.5.

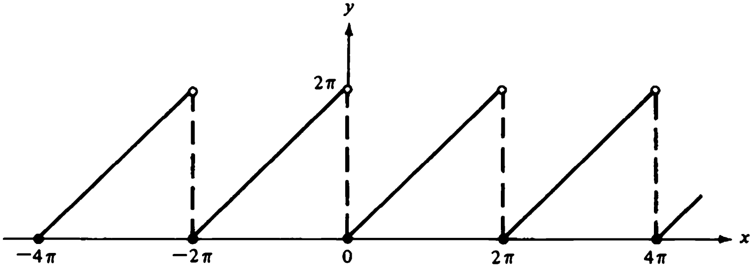


Figure 10.5

The function in this example is different from the function

$$g(x) = x, \qquad -\pi \le x < \pi; \qquad g(x+2\pi) = g(x).$$

In the interval $0 \le x < 2\pi$, the functions f and f' are piecewise continuous with jump discontinuities only at the points 0 and 2π. Therefore, Theorem 1 applies and Figure 10.6 shows the graph of the function to which the Fourier series of f converges. At the points $0, \pm 2\pi, \pm 4\pi, \ldots$, where f is discontinuous, the Fourier series converges to the value π, which is the average value at the jumps. (See Figure 10.6.) At all other points the graphs of f and the function to which its Fourier series converges are identical.

Next, we compute the Fourier series of f. Since the interval $0 \le x < 2\pi$ is not symmetric with respect to the origin, it is advisable to use formulas (4′) and (5′) of Section 10.3 with $c = 0$ and $l = \pi$. The Fourier series of f is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

with

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx \, dx = \frac{1}{\pi}\int_0^{2\pi} x \cos nx \, dx = \begin{cases} 2\pi, & n = 0 \\ 0, & n = 1, 2, 3, \ldots \end{cases}$$

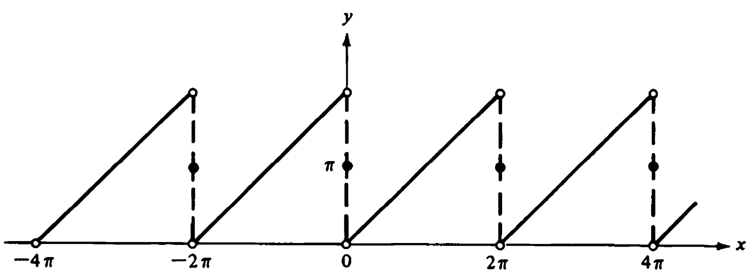


Figure 10.6

and

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx = -\frac{2}{n}, \qquad n = 1, 2, 3, \ldots .$$

Hence,

$$f(x) \sim \pi - \sum_{n=1}^{\infty} \frac{2}{n}\sin nx = \pi - 2\left(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots\right).$$

The symbol $\sim$ can be replaced by the equality sign everywhere except for $x = 0, \pm 2\pi, \pm 4\pi, \ldots$. In particular, in the interval $0 < x < 2\pi$ we obtain the trigonometric identity

$$\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots = \frac{\pi - x}{2},$$

from which, for $x = \pi/2$, we find again

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

EXERCISES

In Exercises 1 through 16, find the Fourier series of the given function. Sketch for a few periods the graph of the function to which the Fourier series converges.

1. $f(x) = x,\ -\pi \le x < \pi;\ f(x+2\pi) = f(x)$

2. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 \le x < \pi \end{cases};\ f(x+2\pi) = f(x)$

3. $f(x) = x^2,\ -1 < x \le 1$

4. $f(x) = x^2,\ 0 \le x < 2\pi;\ f(x + 2\pi) = f(x)$

5. $f(x) = \begin{cases} 1 + x, & -2 \le x < 0 \\ 1 - x, & 0 \le x \le 2 \end{cases};\ f(x+4) = f(x)$

6. $f(x) = |x|,\ -1 \le x < 1;\ f(x+2) = f(x)$

7. $f(x) = \begin{cases} 0, & -\pi < x \le 0 \\ \sin x, & 0 < x \le \pi \end{cases};\ f(x+2\pi) = f(x)$

8. $f(x) = \begin{cases} 0, & -1 < x < 0 \\ x, & 0 \le x < 1 \end{cases}$

9. $f(x) = \begin{cases} 0, & -2 < x \le -1 \\ 1, & -1 < x \le 1 \\ 0, & 1 < x \le 2 \end{cases};\ f(x+4) = f(x)$

10. $f(x) = \begin{cases} -1, & -2 \le x < -1 \\ 0, & -1 \le x < 1 \\ -1, & 1 \le x < 2 \end{cases}$; $f(x+4) = f(x)$

11. $f(x) = \sin^2 x,\ 0 \le x \le 2\pi;\ f(x+2\pi) = f(x)$

12. $f(x) = \sin^2 x,\ -\pi \le x \le \pi;\ f(x+2\pi) = f(x)$

13. $f(x) = \cos^2 x,\ 0 \le x \le 2\pi;\ f(x+2\pi) = f(x)$

14. $f(x) = \cos^2 x,\ -\pi \le x \le \pi;\ f(x+2\pi) = f(x)$

15. $f(x) = \cos 2x,\ 0 \le x \le \pi$

16. $f(x) = \sin 2x,\ 0 \le x \le \pi$

17. Show that

$$\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots = \frac{x}{2},\ -\pi < x < \pi.$$

18. Show that

$$\frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos n\pi x = x^2,\ -1 \le x \le 1.$$

[*Hint:* Use the result of Exercise 3.]

19. Utilize the Fourier series of the function $f(x) = x^2,\ -1 \le x \le 1$, to establish the following results:

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \quad \text{and} \quad \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots.$$

20. Utilize the Fourier series of the function $f(x) = |x|,\ -1 \le x \le 1$, to obtain the following result:

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots.$$

In Exercises 21 through 30, answer true or false.

21. The function $f(x) = \dfrac{1}{x}$ is piecewise continuous in the interval $-\pi \le x \le \pi$.

22. The function $f(x) = \dfrac{1}{x}$ is piecewise continuous in the interval $0 \le x \le \pi$.

23. The Fourier series of the function

$$f(x) = \begin{cases} -3, & -2 < x < 0 \\ 5, & 0 < x < 2 \end{cases}$$

converges to 1 at the points $x = 0, 2$, and -2.

24. The Fourier series of the function

$$f(x) = |x|, \ -\pi \le x < \pi; f(x+2\pi) = f(x)$$

converges to $f(x)$ everywhere.

25. The Fourier series of the function

$$f(x) = x, \ -1 \le x < 1; f(x+2) = f(x)$$

converges to $f(x)$ everywhere.

26. The function

$$f(x) = \sqrt{|x|}, -1 \le x \le 1; f(x+2) = f(x)$$

is continuous everywhere.

27. The function

$$f(x) = \sqrt{|x|}, -1 \le x \le 1; f(x+2) = f(x)$$

satisfies the hypotheses of Theorem 1.

28. The Fourier series of the function

$$f(x) = x^2, 0 \le x < 2\pi; f(x+2\pi) = f(x)$$

does not involve any sine term.

29. The Fourier series of the function

$$f(x) = x^2, \ -\pi \le x \le \pi; f(x+2\pi) = f(x)$$

does not involve any sine terms.

30. The Fourier series of the function

$$f(x) = x^2, \ -\pi \le x \le \pi; f(x+2\pi) = f(x)$$

is equal to $f(x)$ everywhere.

10.5 FOURIER SINE AND FOURIER COSINE SERIES

As we saw in Section 6.2.1 in connection with the solution of the heat equation, sometimes it is necessary to express a given function f as a Fourier series of the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}. \tag{1}$$

In other cases it is necessary to express f as a series of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}. \tag{2}$$

A Fourier series of the form (1) is called a *Fourier sine series*; a Fourier series of the form (2) is called a *Fourier cosine series*. In this section we will show that if a function f is defined in the interval $0 < x < l$, and if f and f' are piecewise continuous there, then in the interval $0 < x < l$ we have the choice to represent f as a Fourier sine series or a Fourier cosine series.

Before we establish the above claim, we will review the concepts of odd and even functions and see how, for such functions, the labor of computing the Fourier coefficients is reduced.

DEFINITION 1

A function f, whose domain is symmetric with respect to the origin, is called even *if $f(x) = f(-x)$ for each x in the domain of f and* odd *if $f(x) = -f(-x)$ for each x in the domain of f.*

For example, the functions $\cos ax$, 1, x^2, $|x|$, $3 - x^2 + x^4 \cos 2x$ are even and $\sin ax$, x, $5x - x^2 \sin 4x$ are odd. Geometrically speaking, a function is even if its graph is symmetric with respect to the y-axis and odd if its graph is symmetric with respect to the origin. The functions whose graphs are sketched in Figure 10.7 are even, and those in Figure 10.8 are odd.

With respect to the operations of addition and multiplication, even and odd functions have the following properties:

(i) even + even = even; (ii) odd + odd = odd;
(iii) even × even = even; (iv) odd × odd = even;
(v) even × odd = odd.

Let us prove, for example, (v). (The others are proved in a similar fashion.) Assume f is even and g is odd. Set $F = fg$. Then

$$F(-x) = f(-x)g(-x) = f(x)\,(-g(x)) = -f(x)g(x) = -F(x),$$

which proves that F is odd.

With respect to integration, even and odd functions have the following useful properties:

$$\int_{-l}^{l} (\text{even})\,dx = 2\int_{0}^{l} (\text{even})\,dx \tag{3}$$

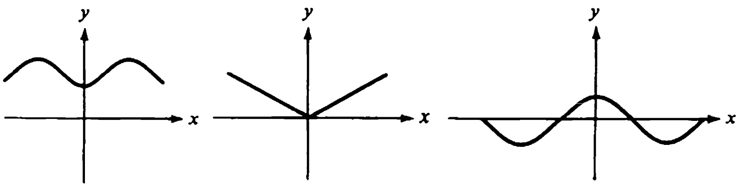


Figure 10.7

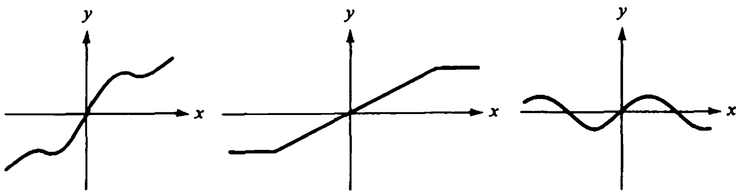


Figure 10.8

and

$$\int_{-l}^{l} (\text{odd})\, dx = 0. \tag{4}$$

We will prove (4). (The proof of (3) is similar.) Assume that f is an odd function. Then

$$\int_{-l}^{l} f(x)\, dx = \int_{-l}^{0} f(x)\, dx + \int_{0}^{l} f(x) dx. \tag{5}$$

Setting $x = -t$, we find

$$\int_{-l}^{0} f(x)\, dx = \int_{l}^{0} f(-t)(-dt) = \int_{l}^{0} -f(-t)\, dt = \int_{l}^{0} f(t)\, dt$$
$$= -\int_{0}^{l} f(t)\, dt.$$

This proves that the right-hand side of (5) is zero, and the proof is complete.

Using properties (3) and (4), the evaluation of the Fourier coefficients

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cos\frac{n\pi x}{l}\, dx, \qquad n = 0, 1, 2, \ldots \tag{6}$$

and

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin\frac{n\pi x}{l}\, dx, \qquad n = 1, 2, 3, \ldots \tag{7}$$

is considerably simplified in the case of even or odd functions.

Even Functions When f is even, the integrand in (6) is even and in (7) is odd. Then, from (3) and (4), we find

$$a_n = \frac{2}{l}\int_{0}^{l} f(x) \cos\frac{n\pi x}{l}\, dx, \qquad n = 0, 1, 2, \ldots \tag{8}$$

and

$$b_n = 0, \qquad n = 1, 2, 3, \ldots.$$

Hence, the Fourier series of an even function is reduced to

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}, \tag{9}$$

where the coefficients a_n are given by (8).

Odd Functions When f is odd, the integrand in (6) is odd and in (7) is even. Then, from (3) and (4), we find

$$a_n = 0, \qquad n = 0, 1, 2, \ldots$$

and

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx, \qquad n = 1, 2, 3, \ldots. \tag{10}$$

Hence, the Fourier series of an odd function is reduced to

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}, \tag{11}$$

where the coefficients b_n are given by (10).

It should be remarked that, for even or odd functions, the formulas for the Fourier coefficients use the values of the function in the interval $0 < x < l$ *only*.

If a function f is defined *only* in the interval $0 < x < l$, we define its *even periodic extension* by

$$g(x) = \begin{cases} f(x), & 0<x<l \\ f(-x), & -l<x<0 \end{cases}; \qquad g(x+2l) = g(x) \tag{12}$$

and its *odd periodic extension* by

$$h(x) = \begin{cases} f(x), & 0<x<l \\ -f(-x), & -l<x<0 \end{cases}; \qquad h(x+2l) = h(x). \tag{13}$$

Note that the functions f, g, and h agree in the interval $0 < x < l$. Furthermore, if f and f' are piecewise continuous on $0 < x < l$, then g and g' and also h and h' are piecewise continuous on $-l < x < l$. Therefore, the hypotheses of Theorem 1 of Section 10.4 are satisfied for the functions g and h. Since g is even, it has a Fourier cosine series which converges to $g(x) = f(x)$ at each point x in the interval $0 < x < l$ where f is continuous; and it converges to the average

$$\frac{g(x-) + g(x+)}{2} = \frac{f(x-) + f(x+)}{2}$$

at each point in $0 < x < l$ where f is discontinuous. Similarly, since h is odd, it has a Fourier sine series which converges to $h(x) = f(x)$ at each point x in $0 < x < l$ where f is continuous, and to the average

$$\frac{h(x-) + h(x+)}{2} = \frac{f(x-) + f(x+)}{2}$$

at each point in $0 < x < l$ where f is discontinuous.

In summary, if a function f is defined *only* in the interval $0 < x < l$, and if f and f' are piecewise continuous there, then f has a Fourier cosine series of the form

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \qquad \text{with } a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx, \qquad n = 0, 1, 2, \ldots \tag{14}$$

and a Fourier sine series of the form

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad \text{with } b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx, \qquad n = 1, 2, 3, \ldots. \tag{15}$$

Furthermore, the convergence of the series in (14) and (15) is as described by Theorem 1 of Section 10.4 for the even and odd periodic extensions of f, respectively.

EXAMPLE 1 Compute the Fourier series of each function:

(a) $f(x) = x^2,\ -\pi < x \le \pi;\ f(x + 2\pi) = f(x)$

(b) $f(x) = x,\ -1 \le x < 1;\ f(x + 2) = f(x)$.

Solution (a) f is an even function and $l = \pi$. Hence, from Equations (9) and (8), we have

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx,$$

with

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx = \begin{cases} \dfrac{2\pi^2}{3}, & n = 0 \\ \dfrac{4(-1)^n}{n^2}, & n = 1, 2, 3, \ldots. \end{cases}$$

Thus,

$$f(x) \sim \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx.$$

(b) f is an odd function and $l = 1$. Hence, from Eqs. (11) and (10), we have

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin n\pi x,$$

with

$$b_n = 2\int_0^1 x\sin\pi x\,dx = -\frac{2(-1)^n}{n\pi}, \qquad n = 1, 2, 3, \ldots.$$

Thus,

$$f(x) \sim -\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\sin n\pi x.$$

EXAMPLE 2 Sketch the even and the odd periodic extension of the function

$$f(x) = \begin{cases} x, 0 < x < 1 \\ 0, 1 < x < 2 \end{cases}.$$

Solution The even periodic extension of f is sketched in Figure 10.9, and the odd periodic extension is sketched in Figure 10.10.

EXAMPLE 3 Compute the Fourier cosine and the Fourier sine series of the function

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}.$$

Sketch the graph of the function to which the Fourier cosine series converges and the graph of the function to which the Fourier sine series converges.

Solution $l = 2$. From (14), the Fourier cosine series of f is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2},$$

with

$$a_n = \int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx = \int_0^1 x\cos\frac{n\pi x}{2}\,dx$$

$$= \begin{cases} \frac{1}{2}, & n = 0 \\ \frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right) + \frac{2}{n\pi}\sin\frac{n\pi}{2}, & n = 1, 2, 3, \ldots . \end{cases}$$

Hencc,

$$f(x) \sim \frac{1}{4} + \sum_{n=1}^{\infty}\left[\frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right) + \frac{2}{n\pi}\sin\frac{n\pi}{2}\right]\cos\frac{n\pi x}{2}.$$

The even periodic extension of f, referred to here as g, is shown graphically in Figure 10.9. From this graph we see that Theorem 1 of Section 10.4 is applicable. The only discontinuities of g are at the points $0, \pm 2, \pm 4, \pm 6, \ldots$ and ± 1,

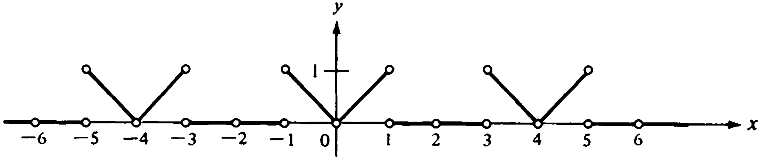


Figure 10.9

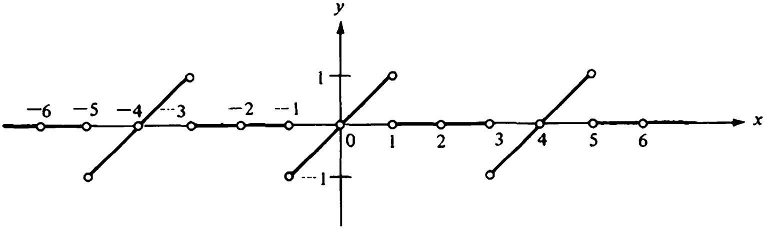


Figure 10.10

$\pm 3, \pm 5, \ldots$. From Figure 10.9 and from Theorem 1 of Section 10.4, we conclude that the Fourier series converges to 0 at the points $\pm 0, \pm 2, \pm 4, \ldots$ and converges to $\frac{1}{2}$ at the points $\pm 1, \pm 2, \pm 3, \ldots$. At every other point x the series converges to $g(x)$. The graph of the function to which the series converges is shown in Figure 10.11 (which is easily constructed from Figure 10.9).

Next we compute the Fourier sine series of f. From (15), with $l = 2$, we have

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2},$$

with

$$b_n = \int_0^2 f(x) \sin \frac{n\pi x}{2}\, dx = \int_0^1 x \sin \frac{n\pi x}{2}\, dx = \frac{4}{n^2\pi^2} \sin \frac{n\pi}{2} - \frac{2}{n\pi} \cos \frac{n\pi}{2}.$$

Hence,

$$f(x) \sim \sum_{n=1}^{\infty} \left(\frac{4}{n^2\pi^2} \sin \frac{n\pi}{2} - \frac{2}{n\pi} \cos \frac{n\pi}{2} \right) \sin \frac{n\pi x}{2}.$$

The odd periodic extension of f, referred to here as h, is shown graphically in Figure 10.10. From this graph we see that Theorem 1 of Section 10.4 is applicable. The only discontinuities of h are at the points $0, \pm 1, \pm 2, \pm 3, \ldots$. The graph of the function to which the series converges is sketched in Figure 10.12.

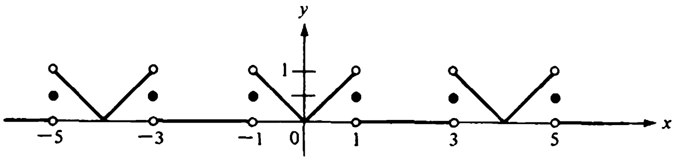


Figure 10.11

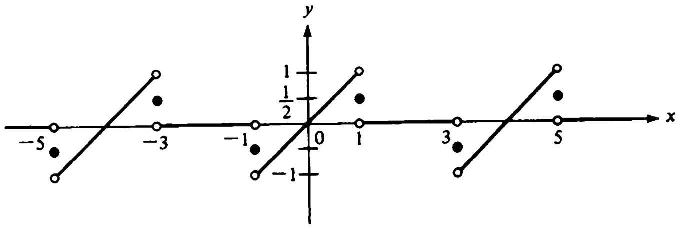


Figure 10.12

EXERCISES

In Exercises 1 through 10, answer true or false.

1. $x^2 \cos x + 2|x|$ is an even function in the interval $-\infty < x < +\infty$.

2. The function

$$F(x) = \begin{cases} x, & 0 < x < 2 \\ -x, & -2 < x < 0 \end{cases}$$

is the odd periodic extension of the function $f(x) = x$, $0 < x < 2$.

3. The function

$$f(x) = x^2, \qquad 0 \le x \le 2; \qquad f(x+2) = f(x)$$

is even.

4. If a function f is odd and is defined at $x = 0$, then $f(0) = 0$.

5. The function

$$f(x) = \begin{cases} 0, & -2 < x < -1 \\ 1, & -1 < x < 1 \\ 0, & 1 < x < 2 \end{cases}; \qquad f(x+4) = f(x)$$

is odd.

6. The function

$$f(x) = \begin{cases} -1, & -2 < x < -1 \\ 1, & -1 < x < 1 \\ -1, & 1 < x < 2 \end{cases}; \qquad f(x+4) = f(x)$$

is even.

7. The Fourier series of the function in Exercise 5 does not contain any sine terms.

8. The Fourier series of the function

$$f(x) = \begin{cases} -1, & -3 < x < 0 \\ 1, & 0 < x < 3 \end{cases}; \quad f(x+6) = f(x)$$

is of the form $\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{3}$ with $b_n = \frac{2}{3}\int_0^3 \sin \frac{n\pi x}{3}\,dx.$

9. The Fourier series of the function

$$f(x) = \begin{cases} -\sin x, & -\pi < x < 0 \\ \sin x, & 0 < x < \pi \end{cases}; \quad f(x+2\pi) = f(x)$$

contains only sine terms.

10. The Fourier series of the function

$$f(x) = \begin{cases} -x, & -2 < x < 0 \\ x, & 0 \le x < 2 \end{cases}$$

is of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2} \quad \text{with} \quad a_n = \int_0^2 x \cos \frac{n\pi x}{2}\,dx, \quad n = 0, 1, 2, \ldots.$$

In Exercises 11 through 18, determine whether the given function is even or odd and utilize this information to compute its Fourier series.

11. $f(x) = |x|,\ -1 \le x < 1; f(x+2) = f(x)$

12. $f(x) = x,\ -\pi \le x < \pi; f(x+2\pi) = f(x)$

13. $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$

14. $f(x) = x^2,\ -1 < x \le 1; f(x+2) = f(x)$

15. $f(x) = x^3,\ -1 \le x \le 1$

16. $f(x) = \begin{cases} -1, & -2 \le x < -1 \\ 1, & -1 \le x < 1 \\ -1, & 1 \le x < 2 \end{cases}; f(x+4) = f(x)$

17. $f(x) = \sin 2x,\ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

18. $f(x) = \begin{cases} 1 + x, & -2 \le x < 0 \\ 1 - x, & 0 \le x \le 2 \end{cases}; \quad f(x + 4) = f(x)$

In Exercises 19 through 28, compute, as indicated, the Fourier sine or Fourier cosine series [see Eqs. (14) and (15)] for the given function. Sketch for a few periods the graph of the function to which the series converges.

19. $f(x) = 1, 0 < x < \pi$; Fourier sine series

20. $f(x) = 1, 0 < x < \pi$; Fourier cosine series

21. $f(x) = x, 0 < x < \pi$; Fourier sine series

22. $f(x) = x, 0 < x < \pi$; Fourier cosine series

23. $f(x) = x^2, 0 < x < 1$; Fourier sine series

24. $f(x) = x^2, 0 < x < 1$; Fourier cosine series

25. $f(x) = \begin{cases} 1, & 0 < x < \dfrac{\pi}{2} \\ 0, & \dfrac{\pi}{2} < x < \pi \end{cases}$; Fourier sine series

26. $f(x) = \begin{cases} 1, & 0 < x < \dfrac{\pi}{2} \\ 0, & \dfrac{\pi}{2} < x < \pi \end{cases}$; Fourier cosine series

27. $f(x) = \begin{cases} x, & 0 < x < \dfrac{1}{2} \\ 0, & \dfrac{1}{2} < x < 1 \end{cases}$; Fourier sine series

28. $f(x) = \begin{cases} x, & 0 < x < \dfrac{1}{2} \\ 0, & \dfrac{1}{2} < x < 1 \end{cases}$; Fourier cosine series

REVIEW EXERCISES

In Exercises 1 through 6, compute the Fourier series of the function.

1. $f(x) = -x^2, \; -3 < x < 3$

2. $f(x) = x, \; -l \le x \le l; \; f(x + 2l) = f(x)$

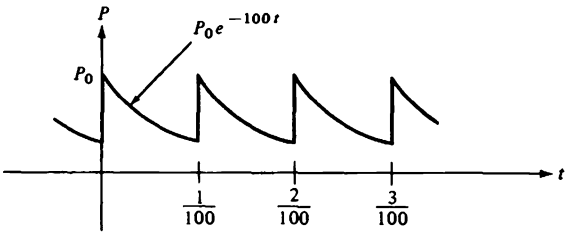
3. $f(x) = \cos \dfrac{x}{3}, \; -\pi < x < \pi; \; f(x + 2\pi) = f(x)$

4. $f(x) = x^3, \; -l \le x \le l; \; f(x + 2l) = f(x)$

5. $f(x) = 1 + e^x, \; -1 \le x < 1$

6. $f(x) = x + |x|, \; -\pi < x \le \pi; f(x + 2\pi) = f(x)$

7. The pressure in a fluid passage[2] has been found to vary, as shown in the figure. In other words, there are regular pulses consisting of a sudden surge followed by an exponential decay occurring at the rate of 100/sec. Find the Fourier series for this periodic wave of pressure versus time.



8. An underdamped harmonic oscillator[3] is subject to a force given by

$$F(t) = \begin{cases} 0, & -\frac{\tau}{2} \le t < 0 \\ F_0, & 0 < t \le \frac{\tau}{2}. \end{cases}$$

Solve for the motion, using the method of Fourier series. [*Hint:* Assume a particular solution of the form

$$y_p(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2n\pi t}{\tau} + b_n \sin \frac{2n\pi t}{\tau}\right).\Bigg]$$

In Exercises 9 through 12, find the Fourier series of the given function. Sketch for a few periods the graph of the function to which the Fourier series converges.

9. $f(x) = \begin{cases} 1, & 0 \le x < \pi \\ -1, & \pi \le x < 2\pi \end{cases} \; ; f(x + 2\pi) = f(x)$

10. $f(x) = 1 + |x|, \; -1 \le x < 1; f(x + 2) = f(x)$

11. $f(x) = 2\cos^2 \frac{x}{2}, \; -\pi \le x \le \pi$

12. $f(x) = \sin 3x, \; 0 \le x \le \frac{2\pi}{3}; f\left(x + \frac{2\pi}{3}\right) = f(x)$

[2]This is Exercise 8 in J. L. Shearer, A. T. Murphy, and H. H. Richardson, *Introduction to Systems Dynamics* (Reading, Mass.: Addison-Wesley Publishing Co., 1971), p. 279. Reprinted with permission of Addison-Wesley Publishing Company.

[3]This is Exercise 3-3.8 in J. Norwood, Jr., *Intermediate Classical Mechanics* (Englewood Cliffs, N.J.: Prentice-Hall, 1979), p. 102. Reprinted by permission of Prentice-Hall, Inc., Englewood Cliffs, New Jersey.

13. Show that, for $-1 \leq x < 1$,

$$\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos n\pi x = \frac{(2|x| - 1)\pi^2}{4}.$$

14. Use the result in Exercise 13 to prove that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

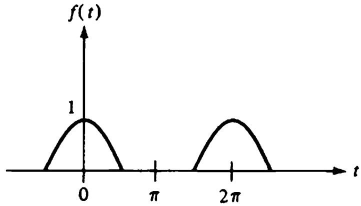
15. Compute the Fourier cosine series of the function

$$f(x) = 1 - x, \qquad 0 < x < \pi.$$

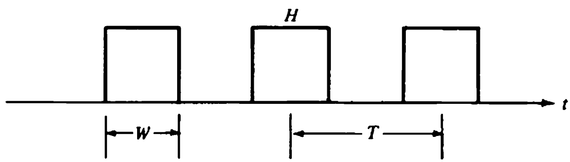
16. Compute the Fourier sine series of the function

$$fx(x) = 1 + x, \qquad 0 < x < \pi.$$

17. Find the Fourier series[4] of the half-wave–rectified cosine shown in the figure.



18. Find the Fourier series expansion[5] of the pulse train shown in the figure. Each pulse has a height H, a duration of W seconds, and a repetition period of T seconds.



[4]This is Exercise c6.4 in R. B. Kerr, *Electrical Network Science* (Englewood Cliffs, N.J.: Prentice-Hall, 1977), p. 335. Reprinted by permission of Prentice-Hall, Inc., Englewood Cliffs, New Jersey.
[5]This is Example 6.4 in ibid., p. 195. Reprinted by permission of Prentice-Hall, Inc., Englewood Cliffs, New Jersey.

CHAPTER 11

An Introduction To Partial Differential Equations

11.1 INTRODUCTION

Partial differential equations are equations that involve partial derivatives of an unknown function. In the case of ordinary differential equations, the unknown function depends on a single independent variable. In contrast, for partial differential equations the unknown function depends on two or more independent variables. In this chapter we present an elementary treatment of partial differential equations. The general theory of partial differential equations is beyond the scope of this book, and we make no attempt to develop it here. Our treatment will focus on some simple cases of first-order equations and some special cases of second-order equations that occur frequently in applications. In this latter category we concentrate on the classical equations of mathematical physics—the heat equation, the potential (Laplace) equation, and the wave equation. The reader interested in the more practical aspects of this subject should concentrate on Sections 11.3, 11.4, and 11.6 through 11.10.

At first glance, a partial differential equation seems to differ from an ordinary differential equation only in that there are more independent variables. However, although the study of partial differential equations frequently utilizes known facts about ordinary differential equations, in general it is quite different. For instance, one of the simplest types of ordinary differential equation is the first-order linear equation (see Section 1.4). The general solution of this ordinary differential equation contains one arbitrary constant; hence this general solution can be interpreted geometrically as a set of two-dimensional curves, each coming about by assigning a different value to the arbitrary constant. Uniqueness of solution (in other words, isolating a specific curve in the set of two-dimensional curves) is accomplished by specifying a two-dimensional point (an initial value $y = y_0$ when $x = x_0$) that the general solution must contain. That is to say, the initial value fixes the value of the arbitrary constant in the general solution. If one considers a first-order linear partial differential equation (to be defined in Section 11.2), the results are very different. Let us assume that the independent variables are x and y and that the unknown function is u. The general solution can be interpreted geometrically as a collection of three-dimensional surfaces. Unfortunately, no *simple* condition serves to isolate (uniquely determine) a specific surface from this collection. Example 1 illustrates this fact.

EXAMPLE 1 Show that the following functions are solutions of the differential equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0.$$

(a) $u = f(x - y)$, where f is *any* function having a continuous derivative; and

(b) $u = (x - y)^n$, where n is any positive integer.

Solution (a) $u = f(x - y) \Rightarrow \dfrac{\partial u}{\partial x} = [f'(x - y)](1)$,

$$\frac{\partial u}{\partial y} = [f'(x - y)](-1).$$

Thus,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = f'(x - y) - f'(x - y) = 0.$$

(b)

$$u = (x - y)^n \Rightarrow \frac{\partial u}{\partial x} = n(x - y)^{n-1}(1),$$

$$\frac{\partial u}{\partial y} = n(x - y)^{n-1}(-1).$$

Thus,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = n(x - y)^{n-1} - n(x - y)^{n-1} = 0.$$

The solution in part (a) demonstrates that we cannot expect to obtain uniqueness of solution by specifying a single point that the general solution must contain. One point cannot uniquely determine the arbitrary function f. More specifically, suppose that $f(x - y) \equiv c(x - y)$, where c is an arbitrary constant. If we specify that the solution surface contain the origin (in other words, $u = 0$ when $x = y = 0$), there is an infinity of solution surfaces (corresponding to the values of c) that contain this point. The solution in part (b), on the other hand, demonstrates that we cannot expect (in general) to obtain uniqueness of solution by specifying a particular curve that the general solution must contain. In particular, if it is specified that the solution surface contain the curve $y = x$ in the xy-plane (in other words, $u = 0$ when $y = x$), then there is an infinity of solution surfaces (corresponding to the values of n) which contain this curve.

Example 1 illustrates that, for partial differential equations, uniqueness of solutions is not (in general) accomplished by simply specifying a point or a curve that the general solution must contain. Appropriate conditions that produce uniqueness of solutions usually depend on the form of the partial differential equation. We will not attempt to investigate all of the ramifications of this issue, but rather we will discuss particular situations.

In spite of the above differences, there are some similarities and analogies between ordinary differential equations and partial differential equations. In Section 11.2 we will discuss a few of them.

EXERCISES

1. Show that (a) $u = f(x + y)$, where f is any function possessing a continuous derivative, and (b) $u = (x + y)^n$, where n is a positive integer are solutions of the partial differential equation $\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 0$. Give an argument to support the statement that there is an infinity of solutions which contain the point $(0, 0, 0)$.
2. Show that $u = f(2x - y)$, where f is any function possessing a continuous derivative, is a solution of the partial differential equation $\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$.
3. If g is any function such that $g(0) = 0$, and if g possesses a continuous derivative, show that $u = g(3x + 2y)$ is a solution of the partial differential equation $2\frac{\partial u}{\partial x} - 3\frac{\partial u}{\partial y} = 0$. Give an argument to support the statement that there is an infinity of solutions which contain the curve $y = -\frac{3}{2}x$ in the xy-plane.

11.2 DEFINITIONS AND GENERAL COMMENTS

To simplify the notation, we restrict our attention to the case of one unknown function, denoted by u, and no more than three independent variables, denoted by x, y, z. With these constraints we can define a partial differential equation to be a relation of the form

$$F(x, y, z, u, u_x, u_y, u_z, u_{xx}, u_{xy}, u_{xz}, u_{yy}, u_{yz}, u_{zz}, u_{xxx}, \ldots) = 0. \tag{1}$$

In Eq. (1) we have used the subscript notation for partial differentiation, that is,

$$u_z = \frac{\partial u}{\partial z}, \; u_{xy} = \frac{\partial^2 u}{\partial x \partial y}, \; u_{xxx} = \frac{\partial^3 u}{\partial x^3}, \text{ and so on.}$$

We always assume that the unknown function u is "sufficiently well behaved" so that all necessary partial derivatives exist and corresponding mixed partial derivatives are equal; for example,

$$u_{xy} = u_{yx};\; u_{xzx} = u_{xxz} = u_{zxx}; \text{ and so on.}$$

Just as in the case of an ordinary differential equation, we define the *order* of the partial differential equation (1) to be the order of the partial derivative of highest order appearing in the equation. Furthermore, we define the partial differential equation (1) to be *linear* if F is linear as a function of the variables u, u_x, u_y, u_z, u_{xx}, . . . ; that is, F is a linear combination of the unknown function and its derivatives. Equation (1) is said to be *quasilinear* if F is linear as a function of the highest-order derivatives.

The following are examples of partial differential equations:

$$u_x + u_y = 3u_z - 2x^2 - 3z \tag{2}$$

$$u_{xx} = x^2 \tag{3}$$

$$5xyu_{xy} - 3zu_y + 2u = 0, \tag{4}$$

$$7u_x + 8u_y + 3u = 2xe^y, \tag{5}$$

$$u_{xz} + uu_y - 4z = 0, \tag{6}$$

$$5u_{xx} - 8u_{xy} + 9u_{yy} + 4u_x - 3u_y + 2u = 0. \tag{7}$$

Equations (2) and (5) are first order and the rest are second order. Equation (6) is quasilinear, and the rest are linear.

With very few exceptions, we will limit our discussion to linear partial differential equations of order one or two. Thus our most general partial differential equation can be written in the form

$$\begin{aligned} &a_1(x, y, z)u_{xx} + a_2(x, y, z)u_{xy} + a_3(x, y, z)u_{xz} \\ &+ a_4(x, y, z)u_{yy} + a_5(x, y, z)u_{yz} + a_6(x, y, z)u_{zz} \\ &+ a_7(x, y, z)u_x + a_8(x, y, z)u_y + a_9(x, y, z)u_z \\ &+ a_{10}(x, y, z)u = f(x, y, z). \end{aligned} \tag{8}$$

In Eq. (8) it is understood that the function f and the coefficients a_i are known, and u is unknown. By a *solution* of Eq. (8) we mean a continuous function u, of the independent variables x, y, z, with continuous first- and second-order partial derivatives, which, when substituted in (8), reduces Eq. (8) to an identity. Thus, in Example 1 of Section 11.1, we demonstrated by direct substitution that $u = f(x - y)$ was a solution of $u_x + u_y = 0$.

If $f(x, y, z) \equiv 0$, the partial differential equation (8) is called *homogeneous*; otherwise it is called *nonhomogeneous* (or inhomogeneous). Note that Eqs. (4) and (7) are homogeneous, and Eqs. (2), (3), and (5) are nonhomogeneous. Equation (6) is nonlinear because it is not of the form of Eq. (8).

If every one of the coefficients a_i is a constant, Eq. (8) is called a partial differential equation with *constant coefficients*. If at least one of the a_i is not a constant, Eq. (8) is called a partial differential equation with *variable coefficients*. Equations (2), (3), (5), and (7) have constant coefficients, and Eq. (4) has variable coefficients.

EXAMPLE 1 Find a solution $u = u(x, y)$ of the partial differential equation

$$u_x = x + y. \tag{9}$$

Solution We begin by integrating Eq. (9) "partially with respect to x" (in other words, we integrate with respect to x, treating the variable y as if it were a constant) to obtain

$$u = \tfrac{1}{2}x^2 + xy + c. \tag{10}$$

We note that the "constant of integration" is denoted by c. In order to verify that u, as given in Eq. (10), is a solution of Eq. (9), we need only substitute this expression for u in Eq. (9). When verifying this, notice that even if c were not a constant but a function of the variable y, such as $f(y)$, then u as given by Eq. (10), with c replaced by $f(y)$, would still be a solution of the partial differential equation (9), since $\dfrac{\partial f(y)}{\partial x} = 0$. We conclude that Eq. (10) is not the most general result possible unless we emphasize that c is to be replaced by an arbitrary function of y. Since we seek the most general form possible for the solution, we write

$$u = \tfrac{1}{2}x^2 + xy + f(y), \tag{11}$$

where f is an arbitrary function of y.

This example illustrates another strong contrast between partial differential equations and ordinary differential equations in that solution (11) contains an *arbitrary function* rather than an arbitrary constant.

EXAMPLE 2 Find a solution $u = u(x, y, z)$ of the partial differential equation

$$u_{xy} = z + x. \tag{12}$$

Solution First we integrate partially with respect to y (treating x and z as constants) to obtain

$$u_x = yz + xy + f_1(x, z),$$

where f_1 is an arbitrary function of the variables x and z. Next we integrate partially with respect to x (treating y and z as constants) to obtain

$$u = xyz + \tfrac{1}{2}x^2y + \int^x f_1(s, z)ds + f_2(y, z),$$

where f_2 is an arbitrary function of the variables y and z. If we set

$$f(x, z) \equiv \int^x f_1(s, z)ds, \qquad g(y, z) \equiv f_2(y, z),$$

then our solution takes the form

$$u = xyz + \tfrac{1}{2}x^2y + f(x, z) + g(y, z), \tag{13}$$

where f is an arbitrary function of x and z, and g is an arbitrary function of y and z. f and g are to have continuous first and second partial derivatives with respect to their arguments.

As in the case of ordinary differential equations, we call the solutions (11) and (13) the *general solution* of Eqs. (9) and (12) respectively. Each specific assignment of the arbitrary function(s) in the general solution gives rise to a *particular solution* of the corresponding partial differential equation. Thus,

$$u = \tfrac{1}{2}x^2 + xy + e^y$$

is a particular solution of Eq. (9) [for the particular choice $f(y) = e^y$], and

$$u = xyz + \tfrac{1}{2}x^2y + z\cos x + ye^z$$

is a particular solution of Eq. (12) [for the particular choices $f(x, z) = z\cos x$ and $g(y, z) = ye^z$].

Even though it is difficult (if not impossible) to make all-inclusive general statements about partial differential equations, the following is a reasonable claim for the partial differential equations to be treated in this text.

The general solution to a linear partial differential equation of order n, for an unknown function depending on s independent variables, involves n arbitrary functions, each of which depends on $(s - 1)$ *variables.* (Not necessarily the same set of $(s - 1)$ variables applies to each arbitrary function.)

Thus, in Example 1, $n = 1$, $s = 2$, and the general solution contains the arbitrary function f, which depends on the single variable y. In Example 2, $n = 2$, $s = 3$, and the general solution contains the two arbitrary functions f and g, f depending on the variables x and z, and g depending on the variables y and z. In Example 1 of Section 11.1, $n = 1$, $s = 2$, and the general solution [given as solution (a)] contains the arbitrary function f, which depends on the *single* variable $x - y$.

EXERCISES

For each of the linear partial differential equations in Exercises 1 through 10, state its order, determine whether it is homogeneous or nonhomogeneous, and determine whether it has constant or variable coefficients.

1. $u_{xyz} - 3u_{yy} + 2zu = 3u_x$

2. $u_y + u_{xx} - u_{yy} = 3x^2 - y^2$

3. $u_{xxyz} + 17u = 0$

4. $u_z + u_{zz} - 5u = 0$

5. $3u_{xx} - 5u_{xy} + 4u_{yy} = 0$

6. $u_x - u_y = 3u$

7. $u_y - 3zu + u_z - u_y = 27y + z^2$

8. $u_{xxyyz} + 5xu_y = 0$

9. $u_z - u_x + u_y = \cos z$

10. $u_y - 4u_{xx} = 0$

In Exercises 11 through 20, assume that u is a function of the two independent variables x and y. Integrate each equation to obtain the general solution.

11. $u_y = 0$

12. $u_x = 0$

13. $u_x = 3x^2 + 4y$

14. $u_y = \sin x - \sin y$

15. $u_{xy} = 0$

16. $u_{xx} = 0$

17. $u_{yy} = 0$

18. $u_{yy} = \cos y + e^x$

19. $u_{xy} = \cos y + e^x$

20. $u_x = \sec^2 y$

In Exercises 21 through 30, assume that u is a function of the three independent variables x, y, and z. Integrate each equation to obtain the general solution.

21. $u_{zx} = y$

22. $u_{yz} = 0$

23. $u_{xxy} = 0$

24. $u_{yyz} = \sec^2 x$

25. $u_{xxyz} = 0$

26. $u_x = x^2 + z$

27. $u_z = y$

28. $u_{yxz} = 2$

29. $u_{zz} = y + 3x$

30. $u_{xx} = \sec^2 x$

31. Verify that the general solution of Exercise 21 conforms to the claim about the form of the general solution made at the end of this section.

32. Verify that the general solution of Exercise 23 conforms to the claim about the form of the general solution made at the end of this section.

33. Verify that the general solution of Exercise 25 conforms to the claim about the form of the general solution made at the end of this section.

34. **Elasticity**. The shearing stress and the normal stress in an elastic body are obtainable from *Airy's stress function*, ϕ, where ϕ is a solution of the partial differential equation

$$\phi_{xxxx} + 2\phi_{xxyy} + \phi_{yyyy} = 0.$$

Classify this equation utilizing all definitions of this section that are appropriate.

11.3 THE PRINCIPLE OF SUPERPOSITION

In this section and the next two sections we outline some general ideas and methods of solution. In some instances the methods will be further illustrated in subsequent sections.

Our partial differential equation has the appearance of Eq. (8), Section 11.2. For simplicity, we write Eq. (8) in the abbreviated form

$$A[u] = f. \tag{1}$$

We speak of the symbol A as an *operator*, and the manner in which the operator A "operates" on the function u (denoted by $A[u]$) is defined by the left-hand side of Eq. (8). Thus, for the operator of Eq. (8) we write,

$$A[u] \equiv a_1(x, y, z)u_{xx} + \cdots + a_9(x, y, z)u_z + a_{10}(x, y, z)u.$$

DEFINITION 1

An operator A is called a linear operator if $A[c_1u_1 + c_2u_2] = c_1A[u_1] + c_2A[u_2]$ for every choice of the constants c_1, c_2 and for every permissable (in other words, $A[u_1]$, $A[u_2]$, and $A[c_1u_1 + c_2u_2]$ make sense) choice of functions u_1 and u_2.

DEFINITION 2

Given the nonhomogeneous partial differential equation (1), *the equation* $A[u] = 0$ *is called the associated homogeneous equation.*

A very important property of linear partial differential equations is contained in the following principle.

Principle of Superposition *Let* $f_1, f_2, \ldots, f_m$ *be any functions and let* $c_1, c_2, \ldots, c_m$ *be any constants. If* A *is a linear operator and if* $u_1, u_2, \ldots, u_m$ *are, respectively, solutions of the equations* $A[u_1] = f_1$, $A[u_2] = f_2, \ldots, A[u_m] = f_m$, *then* $u = c_1u_1 + c_2u_2 + \cdots + c_mu_m$ *is a solution of the equation*

$$A[u] = c_1f_1 + c_2f_2 + \cdots + c_mf_m.$$

Proof Using the linearity of A we have

$$\begin{aligned} A[u] &= A[c_1u_1 + c_2u_2 + \cdots + c_mu_m] \\ &= c_1A[u_1] + c_2A[u_2] + \cdots + c_mA[u_m] \\ &= c_1f_1 + c_2f_2 + \cdots + c_mf_m, \end{aligned}$$

and the proof is complete.

Two important consequences of the principle of superposition are as follows: (i) If $u_1, u_2, \ldots, u_m$ are solutions of $A[u] = 0$ and $c_1, c_2, \ldots, c_m$ are any constants, then $\sum_{i=1}^{m} c_iu_i$ is also a solution of $A[u] = 0$; that is, any linear combination of solutions is a solution. (ii) If u_h is a solution of $A[u] = 0$ and if u_p is a particular solution of $A[u] = f$, then $u = u_h + u_p$ is a solution of $A[u] = f$; that is, the sum of a solution of the homogeneous equation and a particular solution is also a solution. As in the case of ordinary differential equations, if u_h is the general solution of the homogeneous equation $A[u] = 0$, and u_p is a particular solution of $A[u] = f$, then $u = u_h + u_p$ is the general solution of $A[u] = f$.

These two consequences are similar to the manner in which we generated the general solution for a nonhomogeneous linear ordinary differential equation (see Chapter 2). The same approach is sometimes useful in trying to solve a nonhomogeneous linear partial differential equation, but the methods of solution are not as concrete or systematic for partial differential equations as for ordinary differential equations.

EXAMPLE 1 Suppose that u is a function depending on the variables, x, y, and z, and that u satisfies the partial differential equation

$$u_{xy} = z + x. \tag{2}$$

Show that (i) the general solution of the associated homogeneous equation $u_{xy} = 0$ is given by $u_h = f(x, z) + g(y, z)$, where f and g are arbitrary functions; (ii) $u_1 = xyz$ is a particular solution of the equation $u_{xy} = f_1$, where

$f_1(x, y, z) = z$; (iii) $u_2 = \frac{1}{2}x^2y$ is a solution of $u_{xy} = f_2$, where $f_2(x, y, z) = x$; and (iv) $u = u_h + u_1 + u_2$ is the general solution of (2).

Solution

(i) $u_h = f(x, z) + g(y, z) \Rightarrow (u_h)_x = \dfrac{\partial f(x, z)}{\partial x} \equiv h(x, z)$

$$\Rightarrow (u_h)_{xy} = 0,$$

(ii) $u_1 = xyz \Rightarrow (u_1)_x = yz \Rightarrow (u_1)_{xy} = z,$

(iii) $u_2 = \frac{1}{2}x^2y \Rightarrow (u_2)_x = xy \Rightarrow (u_2)_{xy} = x.$

(iv) Note that Eq. (2) is of the form $A[u] = f_1 + f_2$,

where $A[u] = u_{xy} \equiv \dfrac{\partial^2 u}{\partial x \partial y}$. Now

$$\begin{aligned} A[c_1u_1 + c_2u_2] &= \frac{\partial^2}{\partial x \partial y}[c_1u_1 + c_2u_2] \\ &= \frac{\partial^2}{\partial x \partial y}[c_1u_1] + \frac{\partial^2}{\partial x \partial y}[c_2u_2] \\ &= c_1\frac{\partial^2}{\partial x \partial y}[u_1] + c_2\frac{\partial^2}{\partial x \partial y}[u_2] \\ &= c_1A[u_1] + c_2A[u_2]. \end{aligned}$$

Thus A is a linear operator. By the principle of superposition (with $c_1 = c_2 = 1$), we have that

$$u_p = u_1 + u_2 = xyz + \frac{1}{2}x^2y$$

is a particular solution of Eq. (2). Therefore, the general solution of (2) (compare with Example 2, Section 11.2) is given by

$$u = f(x,z) + g(y,z) + xyz + \frac{1}{2}x^2y. \tag{3}$$

EXERCISES

In Exercises 1 through 13, show that the operator A is linear.

1. $A[u] = u_{xx}$

2. $A[u] = u_y$

3. $A[u] = u_{yy}$

4. $A[u] = u_x$

5. $A[u] = u_{xy}$

6. $A[u] = u$

7. $A[u] = u_{xx} + u_{yy}$

8. $A[u] = u_y - cu_{xx}$, c a constant

9. $A[u] = u_{yy} - c^2u_{xx}$, c a constant

10. $A[u] = 3u_x + 4u_y - 7u$

11. $A[u] = u_{xx} + 5u_{xy} - 2u_{yy}$

12. $A[u] = au_{xx} + 2bu_{xy} + cu_{yy}$, a, b, c constants

13. $A[u] = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu$, a, b, c, d, e, f constants

In Exercises 14 through 23, assume that u is a function of the two independent variables x and y. Verify that the given u_h is the general solution of the associated homogeneous equation (f, g, and k denote arbitrary functions of sufficient differentiability for the solution to be meaningful).

14. $u_x = y$; $u_h = f(y)$

15. $u_y = xy$; $u_h = f(x)$

16. $u_{xy} = yx$; $u_h = f(x) + g(y)$

17. $u_{yy} = 2\cos x$; $u_h = yf(x) + g(x)$

18. $u_{xx} = 0$; $u_h = xf(y) + g(y)$

19. $u_{xxy} = x$; $u_h = f(x) + xg(y) + k(y)$

20. $u_{xyy} = xy$; $u_h = yf(x) + g(x) + k(y)$

21. $u_x = y + x^2$; $u_h = f(y)$

22. $u_y = \sin x - e^y$; $u_h = f(x)$

23. $u_{xx} = 2x - 3y$; $u_h = xf(y) + g(y)$

24. Show that $u_p = xy$ is a particular solution of $u_x = y$. Write the general solution for this equation.

25. Show that $u_p = \frac{1}{2}xy^2$ is a particular solution of $u_y = xy$. Write the general solution for this equation.

26. Show that $u_p = \frac{1}{4}x^2y^2$ is a particular solution of $u_{xy} = yx$. Write the general solution of this equation.

27. Show that $u_p = 0$ is a particular solution of $u_{xx} = 0$. Write the general solution of this equation.

28. Show that $u_p = \frac{1}{6}x^3y$ is a particular solution of $u_{xxy} = x$. Write the general solution of this equation.

29. Show that $u_p = \frac{1}{12}x^2y^3$ is a particular solution of $u_{xyy} = xy$. Write the general solution of this equation.

30. Show that $u_p = xy$ is a particular solution of $u_x = y$, and $u_p = \frac{1}{3}x^3$ is a particular solution of $u_x = x^2$. Write the general solution of $u_x = y + x^2$.

31. Show that $u_p = y\sin x$ is a particular solution of $u_y = \sin x$, and $u_p = -e^y$ is a particular solution of $u_y = -e^y$. Write the general solution of $u_y = \sin x - e^y$.

32. Show that $u_p = \frac{1}{3}x^3$ is a particular solution of $u_{xx} = 2x$, and $u_p = -\frac{3}{2}x^2y$ is a particular solution of $u_{xx} = -3y$. Write the general solution of $u_{xx} = 2x - 3y$.

Linear Homogeneous Partial Differential Equations with Constant Coefficients In Section 2.4 we saw that for linear homogeneous ordinary differential equations with constant coefficients, the trial solution $e^{\lambda x}$, λ a constant, led to the requirement that λ be a root of an algebraic polynomial, namely the characteristic polynomial. If we have a linear homogeneous partial differential equation which has x and y as independent variables, we are motivated to assume a trial solution of the form $e^{\lambda x + \mu y}$, where λ and μ are constants to be determined. Thus we substitute $u = e^{\lambda x + \mu y}$ into the equation

$$a_1u_{xx} + a_2u_{xy} + a_3u_{yy} + a_4u_x + a_5u_y + a_6u = 0 \tag{4}$$

to obtain

$$e^{\lambda x + \mu y}[a_1\lambda^2 + a_2\lambda\mu + a_3\mu^2 + a_4\lambda + a_5\mu + a_6] = 0.$$

Consequently $e^{\lambda x + \mu y}$ will be a solution of Eq. (4) if and only if λ, μ satisfy

$$a_1\lambda^2 + a_2\lambda\mu + a_3\mu^2 + a_4\lambda + a_5\mu + a_6 = 0. \tag{5}$$

Note that Eq. (5) is a single algebraic equation[1] in two unknowns, and in general there will be an infinity of solutions (λ, μ).[2] In particular, if (λ_1, μ_1) is a specific solution of Eq. (5), then $e^{\lambda_1 x + \mu_1 y}$ is a particular solution of Eq. (4). By the principle of superposition, it follows that if (λ_1, μ_1), (λ_2, μ_2), . . . , (λ_n, μ_n) are n different solutions of Eq. (5), and if $c_1, c_2, \ldots, c_n$ are arbitrary constants, then $\sum_{i=1}^{n} c_i e^{\lambda_i x + \mu_i y}$ is a solution of Eq. (4).

In Exercises 33 through 48, find a particular solution in the form $u = e^{\lambda x + \mu y}$.

33. $3u_x + u_y + u = 0$

34. $u_x - 2u_y + 5u = 0$

35. $2u_x + 3u_y - 8u = 0$

36. $u_x + u_y - u = 0$

37. $5u_x - 3u_y - 2u = 0$

38. $u_x - u_y + 5u = 0$

39. $u_{xx} + 2u_{xy} + u_{yy} + 3u_x + 3u_y + 2u = 0$

40. $u_{xx} - 2u_{xy} + u_{yy} - 2u_x + 2u_y - 3u = 0$

41. $u_{xx} - 2u_{xy} + u_{yy} + 5u_x - 5u_y + 4u = 0$

42. $u_{xx} + 2u_{xy} + u_{yy} - 5u_x - 5u_y + 6u = 0$

43. $4u_{xx} - 4u_{xy} + u_{yy} + 4u_x - 2u_y - 3u = 0$

44. $u_{xx} + 4u_{xy} + 4u_{yy} + 6u_x + 12u_y + 8u = 0$

[1]We recognize Eq. (5) as representing a conic section in the $\lambda\mu$-plane. Thus the graph is either a hyperbola, a parabola, an ellipse, or a degeneracy of one of these curves.
[2]For motivation see Footnote 1.

45. $4u_{xx} + 12u_{xy} + 9u_{yy} - 2u_x - 3u_y - 6u = 0$

46. $u_{xx} - 10u_{xy} + 25u_{yy} - 3u_x + 15u_y - 10u = 0$

47. $u_{xx} - u_{xy} + u_{yy} - u_x + 2u_y - 2u = 0$

48. $2u_{xx} - u_{xy} - 2u_{yy} - u_x + 2u_y + 2u = 0$

49. To find particular solutions of linear partial differential equations, one can sometimes use an approach similar to that of undetermined coefficients for ordinary differential equations. Thus, if the partial differential equation is of the form

$$A[u] = \sum_{i=1}^{n} f_i,$$

with each f_i a "simple" function, then for each f_i we can find a particular solution by undetermined coefficients and then use the principle of superposition to obtain a particular solution. Find a particular solution for each of the following partial differential equations.

(a) $u_x + u_y - u = 3x^2$ (b) $u_x + u_y + u = 2\cos y$

(c) $u_{xx} + u_{yy} + u_x - u_y + u = 2x^2 - 3y^2$

(d) $u_{xx} - 3u_{xy} + 5u_{yy} = 10e^{3x-4y}$ [*Hint:* Set $u_p = Ae^{3x-4y}$.]

50. Laplacian The operator Δ defined by

$$\Delta[u] = u_{xx} + u_{yy} + u_{zz}$$

is referred to as the three-dimensional *Laplacian operator*. If the term u_{zz} is removed, the resulting operator is called two-dimensional, and if u_{zz} and u_{yy} are removed, the resulting operator is called one-dimensional.

(a) Show that the one-, two-, and three-dimensional Laplacian operators are linear operators.

(b) If we change variables from rectangular coordinates to polar coordinates by the formulas $x = r\cos\theta$, $y = r\sin\theta$, show that the two-dimensional Laplacian operator has the form

$$\Delta[u] = \frac{1}{r}[(ru_r)_r + u_{\theta\theta}/r].$$

[*Hint:* Make repeated use of the chain rule; for example, $u_r = u_x x_r + u_y y_r$.]

(c) If we change variables from rectangular coordinates to spherical coordinates by the formulas $x = r\sin\phi\cos\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\phi$, show that the three-dimensional Laplacian operator has the form

$$\Delta[u] = \frac{1}{r^2}(r^2u_r)_r + \frac{1}{r^2\sin\phi}(\sin\phi\, u_\phi)_\phi + \frac{1}{r^2\sin^2\phi}u_{\theta\theta}.$$

[*Hint:* Make repeated use of the chain rule; for example, $u_\phi = u_x x_\phi + u_y y_\phi + u_z z_\phi$.]

Remark The classical equations of mathematical physics are

$$u_{tt} = c^2\Delta[u] \qquad \text{(the wave equation);}$$

$$u_t = a\Delta[u] \qquad \text{(the heat equation);}$$

$$0 = \Delta[u] \qquad \text{(the potential or Laplace equation).}$$

In each case the equation is called one-, two-, or three-dimensional, depending on whether the Laplacian operator is one-, two-, or three-dimensional. The one-dimensional wave equation is discussed in Section 11.6; the one-dimensional heat equation is discussed in Section 11.7; and the two-dimensional Laplace equation is discussed in Section 11.8.

11.4 SEPARATION OF VARIABLES

A frequently used method for finding solutions to linear homogeneous partial differential equations is known as *separation of variables*. In this method we try to write the solution as a product of functions, each of which depends on exactly one of the independent variables. For example, we would try to write the solution of the partial differential equation

$$u_{xx} + u_{yy} + u_{zz} = 0$$

in the form $u = X(x)Y(y)Z(z)$, where the functions X, Y, and Z are to be determined. For the partial differential equation

$$u_{xx} - u_{yy} = 0,$$

we would try to write the solution in the form $u = X(x)Y(y)$, where the functions X, Y are to be determined. The basic ideas and manipulations involved in the method are illustrated in the following examples.

EXAMPLE 1 For the partial differential equation

$$u_{xx} - u_{yy} = 0, \tag{1}$$

find a solution in the form $u = X(x)Y(y)$.

Solution If

$$u = X(x)Y(y),$$

then

$$u_x = X'(x)Y(y),\ u_{xx} = X''(x)Y(y),\ u_y = X(x)Y'(y),\ u_{yy} = X(x)Y''(y).$$

Substitution of these results in Eq. (1) leads to

$$X''(x)Y(y) - X(x)Y''(y) = 0.$$

Divide this latter equation by $u = X(x)Y(y)$ (assuming that $u \neq 0$) to obtain

$$\frac{X''(x)}{X(x)} - \frac{Y''(y)}{Y(y)} = 0$$

or

$$\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)}. \tag{2}$$

Since $\frac{X''(x)}{X(x)}$ does not contain the variable y, we note that changes in y will not have any effect on the expression $\frac{X''(x)}{X(x)}$. Thus, if (2) is to be an equality, it must happen that changes in the variable y do not affect the expression $\frac{Y''(y)}{Y(y)}$ either. Similarly, changes in x should not affect the expression $\frac{X''(x)}{X(x)}$. The net conclusion is that in order for (2) to be an equality, the expressions $\frac{X''(x)}{X(x)}$ and $\frac{Y''(y)}{Y(y)}$ must be constants. In fact, they must be the same constant. If the constant is denoted by λ, we can write

$$\frac{X''(x)}{X(x)} = \lambda$$

and

$$\frac{Y''(y)}{Y(y)} = \lambda.$$

Thus,

$$X''(x) - \lambda X(x) = 0 \tag{3}$$

and

$$Y''(y) - \lambda Y(y) = 0. \tag{4}$$

Equations (3) and (4) are ordinary differential equations with constant coefficients and can be solved by the methods of Section 2.5 to yield.

$$X(x) = \begin{cases} c_1 e^{\sqrt{\lambda} x} + c_2 e^{-\sqrt{\lambda} x}, & \lambda > 0 \\ c_1 + c_2 x, & \lambda = 0 \\ c_1 \cos \sqrt{-\lambda} x + c_2 \sin \sqrt{-\lambda} x, & \lambda < 0\ ; \end{cases}$$

$$Y(y) = \begin{cases} c_3 e^{\sqrt{\lambda} y} + c_4 e^{-\sqrt{\lambda} y}, & \lambda > 0 \\ c_3 + c_4 y, & \lambda = 0 \\ c_3 \cos \sqrt{-\lambda} y + c_4 \sin \sqrt{-\lambda} y, & \lambda < 0\ . \end{cases}$$

Thus,

$$u = X(x)Y(y) = \begin{cases} (c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x})(c_3 e^{\sqrt{\lambda}y} + c_4 e^{-\sqrt{\lambda}y}), & \lambda > 0 \\ (c_1 + c_2 x)(c_3 + c_4 y), & \lambda = 0 \\ (c_1 \cos\sqrt{-\lambda}x + c_2 \sin\sqrt{-\lambda}x) \cdot \\ \quad (c_3 \cos\sqrt{-\lambda}y + c_4 \sin\sqrt{-\lambda}y), & \lambda < 0. \end{cases}$$

Without further information we have no way of knowing the value of λ; hence we cannot specify the form of the solution. In many practical problems there are other conditions that the solution must satisfy; these conditions usually dictate the value of λ and the form of the solution (see Sections 11.6–11.10).

EXAMPLE 2 For the partial differential equation

$$3u_x - 2u_y - 5u_z = 0,$$

find a solution in the form $u = X(x)Y(y)Z(z)$.

Solution If

$$u = X(x)Y(y)Z(z)$$

then

$$u_x = X'(x)Y(y)Z(z), \qquad u_y = X(x)Y'(y)Z(z), \qquad \text{and} \qquad u_z = X(x)Y(y)Z'(z).$$

Substitution into the partial differential equation yields

$$3X'(x)Y(y)Z(z) - 2X(x)Y'(y)Z(z) - 5X(x)Y(y)Z'(z) = 0.$$

Dividing by $u = X(x)Y(y)Z(z)$ (assuming $u \neq 0$), we have

$$\frac{3X'(x)}{X(x)} - \frac{2Y'(y)}{Y(y)} - \frac{5Z'(z)}{Z(z)} = 0$$

or

$$\frac{3X'(x)}{X(x)} = \frac{2Y'(y)}{Y(y)} + \frac{5Z'(z)}{Z(z)}. \tag{5}$$

Using the same type of argument as in Example 1, we conclude that the only way that Eq. (5) can be an equality is that both sides of the equation equal a constant, say λ. Thus

$$\frac{3X'(x)}{X(x)} = \lambda \tag{6}$$

and

$$\frac{2Y'(y)}{Y(y)} + \frac{5Z'(z)}{Z(z)} = \lambda. \tag{7}$$

Equation (6) has as general solution $X(x) = c_1e^{(\lambda/3)x}$. Equation (7) can be rewritten as

$$\frac{2Y'(y)}{Y(y)} = \lambda - \frac{5Z'(z)}{Z(z)}. \tag{8}$$

Once more we argue that in order for (8) to be an equality, we must have

$$\frac{2Y'(y)}{Y(y)} = \mu, \tag{9}$$

and

$$\lambda - \frac{5Z'(z)}{Z(z)} = \mu, \tag{10}$$

where μ is a constant. The solution to Eq. (9) is $Y(y) = ce^{(\mu/2)y}$, and the solution to Eq. (10) is $Z(z) = c_3e^{[(\lambda-\mu)/5]z}$. Therefore a solution to the partial differential is

$$u = (c_1e^{(\lambda/3)x})\,(c_2e^{(\mu/2)y})\,(c_3e^{[(\lambda-\mu)/5]z}) = ke^{(\lambda/3)x+(\mu/2)y+[(\lambda-\mu)/5]z}.$$

In Examples 1 and 2, the partial differential equations have constant coefficients, but the method can also be applied to equations with variable coefficients. It is not the case, however, that every linear homogeneous partial differential equation can be solved by the method of separation of variables. There are many equations for which the method does not apply. Example 3 is a case in point.

EXAMPLE 3 Show that the variables "do not separate" for the partial differential equation

$$u_{xy} + u_{xx} + u = 0.$$

Solution We try a solution in the form $u = X(x)Y(y)$. Then

$$u_x = X'(x)Y(y),\ u_{xy} = X'(x)Y'(y),\ u_{xx} = X''(x)Y(y).$$

Substitution of these results into the partial differential equation leads to

$$X'(x)Y'(y) + X''(x)Y(y) + X(x)Y(y) = 0.$$

It is not possible to algebraically manipulate this latter equation to a form $P(x) = Q(y)$, therefore we conclude that the method does not work for this partial differential equation; that is, the variables "do not separate."

EXERCISES

In Exercises 1 through 22, assume a solution in the form $u = X(x)Y(y)$. Show that the equation "separates," and find the differential equations that X and Y must satisfy.

1. $u_x - 3u_y = 0$

2. $4u_x + 3u_y = 0$

3. $2u_x + 5u_y = 0$

4. $7u_x - 6u_y = 0$

5. $u_x + u_y + u = 0$

6. $2u_x - 3u_y - u = 0$

7. $u_y - u_{xx} = 0$

8. $u_y - cu_{xx} = 0$, c a constant

9. $u_{yy} - u_{xx} = 0$

10. $u_{yy} - c^2u_{xx} = 0$, c a constant

11. $u_{xx} + u_{yy} + u_x + u_y = 0$

12. $u_{xx} + u_{yy} + u_x - u_y = 0$

13. $u_{xx} - u_{yy} - u_x + u_y = 0$

14. $u_{xx} - u_{yy} - u_x - u_y = 0$

15. $u_y - u_{xx} + u_x = 0$

16. $u_{xx} + u_{yy} + u_x + u_y + u = 0$

17. $3u_y - 5u_{xx} - 6u_x = 0$

18. $5u_{xx} - 6u_{yy} + u_x - 3u_y + u = 0$

19. $2u_{xx} + 3u_{yy} + u = 0$

20. $au_{xx} + cu_{yy} + du_x + eu_y = 0$, a, c, d, e, constants

21. $au_{xx} + cu_{yy} + du_x + eu_y + fu = 0$, a, c, d, e, f, constants

22. $a(x)u_{xx} + c(y)u_{yy} + d(x)u_x + e(y)u_y + fu = 0$, $a(x)$, $d(x)$ continuous functions of x; $c(y)$, $e(y)$ continuous functions of y, f a constant

In Exercises 23 through 32, assume a solution in the form $u = X(x)Y(y)Z(z)$. Show that the equation "separates" and find the differential equations that X, Y, and Z must satisfy.

23. $u_x - u_y + u_z = 0$

24. $2u_x + 3u_y + 4u_z = 0$

25. $u_x + 2u_y - 2u_z = 0$

26. $u_x + u_y + u_z + u = 0$

27. $u_{xx} + u_{yy} + u_{zz} = 0$

28. $u_y - u_{xx} - u_{zz} = 0$

29. $u_{yy} - u_{xx} - u_{zz} = 0$

30. $u_{yy} - 3u_{xx} - 3u_{zz} = 0$

31. $2u_{xx} - u_{yy} + u_{zz} + u = 0$

32. $u_{xx} - u_{yy} + u_z - u = 0$

In Exercises 33 through 39, assume a solution in the form $u = X(x)Y(y)$. Show that the equation does not "separate."

33. $(y + x)u_x + u_y = 0$

34. $u_{xy} + (2x + 3y)u_y = 0$

35. $u_x + f(x, y)u_y = 0$, where $\dfrac{\partial f}{\partial x} \neq 0$, $\dfrac{\partial f}{\partial y} \neq 0, f(x, y) \neq f_1(x) \cdot f_2(y)$

36. $u_{xx} + u_{xy} - 2u = 0$

37. $u_{xx} + xu_{yy} - u_y = 0$

38. $u_{yy} + xu_y - u = 0$

39. $u_{xx} - u_{yy} + 3xu_x - 3xu_y = 0$

40. Show that if f is an arbitrary twice differentiable function, and λ is a constant, then $u = f(y + \lambda x)$ is a solution of

$$a_1u_{xx} + a_2u_{xy} + a_3u_{yy} = 0, \tag{11}$$

if and only if λ is a solution of

$$a_1\lambda^2 + a_2\lambda + a_3 = 0. \tag{12}$$

Thus, if λ_1 and λ_2 ($\lambda_1 \neq \lambda_2$) are the roots of Eq. (12), then the general solution of Eq. (11) is given by $u = f_1(y + \lambda_1 x) + f_2(y + \lambda_2 x)$, where f_1 and f_2 are arbitrary twice differentiable functions. If $\lambda_1 = \lambda_2$, then the general solution is of the form

$$u = f_1(y + \lambda_1 x) + x f_2(y + \lambda_1 x),$$

which can be verified by direct substitution.

DEFINITION
The partial differential equation

$$au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = 0$$

is called hyperbolic *if* $b^2 - ac > 0$, parabolic *if* $b^2 - ac = 0$, elliptic *if* $b^2 - ac < 0$.[2]

In Exercises 41 through 60, classify the second-order partial differential equation as hyperbolic, parabolic, or elliptic. Find the general solution in each case. Refer to Exercise 40.

41. $u_{xx} + 6u_{xy} + 12u_{yy} = 0$

42. $u_{xx} + 20u_{xy} + 64u_{yy} = 0$

43. $5u_{xx} + 10u_{xy} + 20u_{yy} = 0$

44. $6u_{xx} + 4u_{xy} + u_{yy} = 0$

45. $u_{xx} + 9u_{xy} + 4u_{yy} = 0$

46. $u_{xx} + 2u_{xy} + u_{yy} = 0$

47. $u_{xx} + 5u_{xy} + u_{yy} = 0$

48. $4u_{xx} + 8u_{xy} + 4u_{yy} = 0$

49. $u_{xx} + 8u_{xy} + 16u_{yy} = 0$

50. $u_{xx} + u_{xy} + u_{yy} = 0$

51. $6u_{xx} - u_{xy} - u_{yy} = 0$

52. $36u_{xx} + 13u_{xy} + u_{yy} = 0$

53. $21u_{xx} - 10u_{xy} + u_{yy} = 0$

54. $u_{xx} + 2u_{xy} + 5u_{yy} = 0$

55. $u_{xx} + 4u_{xy} + 5u_{yy} = 0$

56. $4u_{xx} - 4u_{xy} + u_{yy} = 0$

57. $u_{xx} - 3u_{xy} - 4u_{yy} = 0$

58. $10u_{xx} + 7u_{xy} + u_{yy} = 0$

59. $u_{xx} + 6u_{xy} + 9u_{yy} = 0$

60. $u_{xx} - u_{yy} = 0$

61. Quantum Mechanics: Helmholtz's Equation In Section 2.8.1 we introduced the Schrödinger wave equation of quantum mechanics, namely

$$\frac{ih}{2\pi}\phi_t = -\frac{h^2}{8m\pi^2}(\phi_{xx} + \phi_{yy} + \phi_{zz}) + V(x, y, z)\phi.$$

(a) Set $\phi(x, y, z, t) = e^{-(i2\pi Et)/h}\, u(x, y, z)$, where E is a constant. Show that u satisfies

$$u_{xx} + u_{yy} + u_{zz} + \frac{8m\pi^2}{h}[E - V(x, y, z)]u = 0. \tag{13}$$

(b) If we set $V \equiv 0$ in Eq. (13), the resulting equation is called *Helmholtz's* equation. For Helmholtz's equation set $u = X(x)Y(y)Z(z)$ and determine the differential equations for X, Y, and Z.

[2]See Footnote 1, page 431.

62. Acoustics The nonlinear partial differential equation

$$(u_x)^{n+1}u_{tt} = a^2u_{xx}$$

occurs in the study of the propagation of sound in a medium. a is a constant and represents the velocity of sound in the medium. Set $u = X(x)T(t)$ and determine the differential equations for X and T.

63. Supersonic Fluid Flow In the study of the supersonic flow of an ideal compressible fluid past an obstacle, the velocity potential satisfies the equation

$$(M^2 - 1)u_{xx} - u_{yy} = 0,$$

where $M(>1)$ is a constant known as the *Mach number* of the flow. Set $u = X(x)Y(y)$ and determine the differential equations for X and Y.

64. Isentropic Fluid Flow The second-order linear partial differential equation

$$u_{xy} - \frac{\alpha}{x + y}(u_x + u_y) = 0$$

occurs in the one-dimensional isentropic flow of a compressible fluid.[3] α is a constant which depends on the fluid.

(a) Set $u = X(x)Y(y)$ and determine the differential equations for X and Y.

(b) Solve the differential equations of part (a).

65. Given

$$u_{tt} + a^2u = u_{xx},$$

where a^2 is a constant. Set $u = X(x)T(t)$ and determine the differential equations for X and T. Solve these differential equations.

66. Acoustics In the study of the transmission of sound through a moving fluid, one considers the velocity potential $u(x, y, z, t)$. (The term *potential* is usually used in physics to describe a quantity whose gradient furnishes a field of force, where gradient F is the vector $[F_x, F_y, F_z]$. In this case the gradient of u yields the velocity of the flow.) It can be shown that u satisfies the three-dimensional wave equation (see the remark following Exercise 50, Section 11.3)

$$u_{xx} + u_{yy} + u_{zz} = \frac{1}{c^2}u_{tt},$$

where the constant c represents the velocity of sound in the medium. Set $u = X(x)Y(y)Z(z)T(t)$ and determine the differential equations for X, Y, Z, and T.

67. Separation of variables for partial differential equations normally relates to the method described in this section. There are other techniques that may also be called separation of variables. To illustrate, let us assume a solution of the equation

[3] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, vol. 2 (New York: Interscience Publishers, 1962), p. 459.

$$u_x + u_y = 0$$

in the form

$$u(x, y) = X(x) + Y(y). \tag{14}$$

X and Y satisfy the differential equations $X' = \lambda$ and $Y' = -\lambda$, respectively, therefore $u = \lambda(x - y) + c$. Find a solution of the form (14) for the partial differential equation

$$(u_x)^2 + (u_y)^2 = 1,$$

where u_x and u_y are assumed to be positive.

68. Magnetic Field Intensity in a Solenoid When a long copper rod is wound with a coil of wire and excited by a current, the following partial differential equation results[4] for the magnetic field intensity H:

$$rH_{rr} + H_r = \frac{4\pi r}{\rho} H_t,$$

where t is time, r is measured from the axis of the copper rod, and ρ is a constant known as the *resistivity* of copper. (a) Set $H = R(r)T(t)$ and determine the differential equations for R and T (call the separation constant $-\lambda$ instead of λ). (b) Find T. (c) Show that the differential equation for R is a special case of Bessel's equation (Section 5.1). [*Hint:* Make the change of variable $r = \frac{1}{\sqrt{\lambda}} x$.]

11.5 INITIAL-BOUNDARY VALUE PROBLEMS: AN OVERVIEW

In Sections 11.6 through 11.10 we consider special cases of the following general problem.

PROBLEM 1

Consider the second-order linear partial differential equation

$$a_1(x, y)u_{xx} + a_2(x, y)u_{xy} + a_3(x, y)u_{yy} + a_4(x, y)u_x + a_5(x, y)u_y$$
$$+ a_6(x, y)u = F(x, y), \qquad 0 < x < l, \qquad 0 < y < m \tag{1}$$

and the conditions

$$A_1u(0, y) + A_2u_x(0, y) = f_1(y), \qquad 0 < y < m \tag{2}$$

$$A_3u(l, y) + A_4u_x(l, y) = f_2(y), \qquad 0 < y < m \tag{3}$$

$$A_5u(x, 0) = f_3(x), \qquad 0 < x < l \tag{4}$$

$$A_6u_y(x, 0) = f_4(x), \qquad 0 < x < l, \tag{5}$$

[4] H. W. Reddick and F. H. Miller, *Advanced Mathematics for Engineers*, 3rd ed. (New York: John Wiley & Sons, 1960), p. 307.

where $a_1(x, y), \ldots, a_6(x, y), f_1(y), f_2(y), f_3(x), f_4(x)$, and $F(x, y)$ are assumed to be known functions and $A_1, \ldots, A_6$ are known constants. l or m, or both, may be infinite.

Is there a function u of two variables x and y that satisfies the partial differential equation (1) and each of the conditions (2)–(5)? Such a problem is referred to as an *initial-boundary value problem* (I-BVP). Equations (2) and (3) are called the *boundary conditions,* and Eqs. (4) and (5) are called the *initial conditions.* For a given problem, one or more of the boundary conditions, or one or more of the initial conditions may be missing.

In Problem 1, if we set F, f_1, f_2, f_3, and f_4 each equal to zero, we obtain the *associated homogeneous problem.* In Sections 11.6–11.8 we demonstrate, for specific cases, how to solve Problem 1 with $F(x, y) = 0$. In Sections 11.9 and 11.10 we consider special cases of Problem 1 with $F(x, y) \neq 0$. While we do not solve the general case of Problem 1, we do provide the necessary ingredients for the solution of many problems that fall into the category of Problem 1. The main prerequisite for utilizing the method we discuss is that the method of separation of variables be applicable to the partial differential equation.

11.6 THE HOMOGENEOUS ONE-DIMENSIONAL WAVE EQUATION: SEPARATION OF VARIABLES

Consider that we have a string that is perfectly flexible (that is to say, the string is capable of transmitting tension but will not transmit bending or shearing forces) and that its mass per unit length is a constant. The string is to be stretched and attached to two fixed points on the x-axis, $x = 0$ and $x = l$. The string is then given an initial displacement and/or an initial velocity parallel to the y-axis, thus setting it in motion. The distance along the string will be denoted by s and as usual

$$ds = \sqrt{(dx)^2 + (dy)^2}.$$

Thus,

$$\frac{\partial s}{\partial x} = \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2}\ .$$

If we assume that the displacement, y, is small enough so that $\left(\frac{\partial y}{\partial x}\right)^2$ is a very small quantity (in comparison to 1), then approximately $\frac{\partial s}{\partial x} = 1$; that is, the length of the string is approximately unchanged (since $s \approx x$). Consequently, the tension in the string is approximately constant. If no other forces are acting on the string, we have the situation illustrated in Figure 11.1, where T represents the (constant) force due to tension.

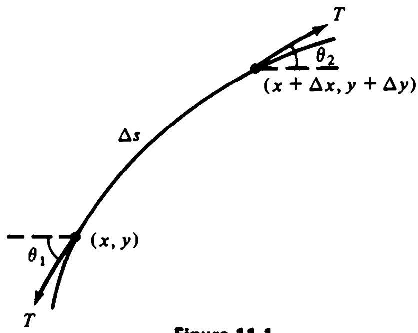


Figure 11.1

The vertical component of the force T at the point (x, y) is given by $-T \sin \theta_1 = -T \dfrac{\partial y}{\partial s}$; however,

$$-T\frac{\partial y}{\partial s} = -T\frac{\partial y}{\partial x}\frac{\partial s}{\partial x} = -T\frac{\partial y}{\partial x}.$$

At the point $(x + \Delta x, y + \Delta y)$, the vertical component of the force T is $T \sin \theta_2$. Since the displacement is considered to be small, $\sin \theta_2 \approx \tan \theta_2 = y'(x + \Delta x)$. Expanding $y'(x + \Delta x)$ in a Taylor's series, the vertical component is

$$T\frac{\partial y}{\partial x} + T\frac{\partial^2 y}{\partial x^2}\Delta x + R,$$

where the remainder, R, has the property

$$\lim_{\Delta x \to 0} \frac{1}{\Delta x} \cdot R = 0.$$

If ρ represents the mass per unit length, then the mass of the portion of string in Figure 11.1 is given by $\rho\Delta s$, which is approximately $\rho\Delta x$. Thus, for motion in the vertical direction, we have by Newton's Second Law of Motion

$$\begin{aligned}\rho\Delta x\frac{\partial^2 y}{\partial t^2} &= T\frac{\partial y}{\partial x} + T\frac{\partial^2 y}{\partial x^2}\Delta x + R + \left(-T\frac{\partial y}{\partial x}\right)\\ &= T\frac{\partial^2 y}{\partial x^2}\Delta x + R.\end{aligned}$$

Dividing by Δx and taking the limit as Δx tends to zero, we obtain the equation of motion of the vertical displacement of the string, namely,

$$\rho\frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2}.$$

Conforming now to our convention of denoting the unknown function by u, we rewrite the equation of motion in the form

$$u_{tt} - c^2u_{xx} = 0, \qquad 0 < x < l, \qquad t > 0, \tag{1}$$

where $c^2 = T/\rho$ is a constant according to our assumptions.

For obvious reasons, Eq. (1) is frequently referred to as the equation of the vibrating string. It is also customary to call Eq. (1) the *one-dimensional wave equation.*

The wave equation is one of three partial differential equations known as the classical equations of mathematical physics. The other two, the potential (Laplace) equation and the heat equation, are discussed in Sections 11.8 and 11.7 respectively. Note that the wave equation is an example of a *hyperbolic* partial differential equation.

A typical initial-boundary value problem for the one-dimensional wave equation is the following.

$$u_{tt} - c^2u_{xx} = 0, \quad 0 < x < l, \quad t > 0 \tag{2}$$

$$u(x, 0) = f(x), \quad 0 < x < l \tag{3}$$

$$u_t(x, 0) = g(x), \quad 0 < x < l \tag{4}$$

$$u(0, t) = 0, \quad t \geq 0 \tag{5}$$

$$u(l, t) = 0, \quad t \geq 0. \tag{6}$$

It can be shown that if f and g satisfy the Dirichlet conditions (see Theorem 1, Section 10.4), the initial-boundary value problem (2)–(6) has a unique solution.

Conditions (5) and (6) can be interpreted as indicating that the ends of the string are attached ("tied") to the x-axis for all time. Condition (3) represents the initial position of the string and condition (4) is the initial velocity.

A standard approach to solving this initial-boundary value problem is to use the separation of variables method of Section 11.4 and Fourier series (Chapter 10). To this end we assume a solution of Eq. (2) to be of the form

$$u(x, t) = X(x)T(t), \tag{7}$$

where X, T are unknown functions to be determined. Substitution of (7) into Eq. (2) yields

$$XT'' - c^2X''T = 0.$$

Thus,

$$\frac{T''}{T} = \frac{c^2X''}{X} = \lambda,$$

with λ a constant. Consequently we have two separate problems:

$$\frac{T''}{T} = \lambda \tag{8}$$

and

$$\frac{c^2X''}{X} = \lambda. \tag{9}$$

The boundary condition (5) demands that $X(0)T(t) = 0$ for all $t \geq 0$; thus, $X(0) = 0$. Similarly, the boundary condition (6) indicates that $X(l) = 0$. The function X, then, is to be a solution of the eigenvalue problem

$$X'' - \frac{\lambda}{c^2}X = 0,$$

$$X(0) = X(l) = 0.$$

This problem can be solved by the methods of Section 6.2, to yield the eigenvalues

$$\lambda_n = -\frac{n^2\pi^2c^2}{l^2}, \qquad n = 1, 2, 3, \ldots \tag{10}$$

and the corresponding eigenfunctions

$$X_n(x) = \sin\frac{n\pi x}{l}, \qquad n = 1, 2, 3, \ldots. \tag{11}$$

With λ given by Eq. (10), Eq. (8) takes the form

$$T'' + \frac{n^2\pi^2c^2}{l^2}T = 0.$$

Hence

$$T_n(t) = a_n\cos\frac{n\pi c}{l}t + b_n\sin\frac{n\pi c}{l}t, \qquad n = 1, 2, 3, \ldots, \tag{12}$$

where a_n and b_n are the integration constants in the general solution.

We conclude that for each specific value of n ($n = 1, 2, 3, \ldots$), the function $X_n(x)T_n(t)$ is a solution of Eq. (2) that satisfies conditions (5) and (6). What about conditions (3) and (4)? Let us investigate condition (3) when $u(x, t) = X_n(x)T_n(t)$, with n not specified but otherwise considered fixed.

$$u(x, 0) = f(x) \Rightarrow X_n(x)T_n(0) = f(x) \Rightarrow \left(\sin\frac{n\pi x}{l}\right)a_n = f(x). \tag{13}$$

The only way that (13) can be satisfied is that $f(x)$ be restricted to be of the form $A\sin\frac{n\pi x}{l}$, where A is a constant. If f is of this form, condition (4) demands that

$$\left(\sin\frac{n\pi x}{l}\right)\left(\frac{n\pi c}{l}b_n\right) = g(x). \tag{14}$$

This, too, restricts g to be of the form $B\sin\frac{n\pi x}{l}$, where B is a constant.

Conditions (13) and (14) place too great a restriction on the permissible forms for f and g; therefore, we consider an alternate approach.

Since $X_n(x)T_n(t)$ is a solution of Eq. (2) for each value of n ($n = 1, 2, 3, \ldots$), and since Eq. (2) is a linear partial differential equation, it seems reasonable to expect that $\sum_{n=1}^{\infty} X_n(x)T_n(t)$ is a solution of Eq. (2). Naturally there is the question of whether or not this infinite series converges. We will not investigate this question here but rather emphasize the method of solution. We consider

$$u(x, t) = \sum_{n=1}^{\infty} X_n(x)T_n(t), \tag{15}$$

with X_n given by (11) and T_n given by (12), to be a solution of Eq. (2). $u(x, t)$ satisfies conditions (5) and (6), as is easily verified. $u(x, t)$ will satisfy condition (3) provided that

$$\sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} = f(x),$$

that is, that the Fourier sine series for $f(x)$ in the interval $0 \leq x \leq l$ be $\sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}$. Consequently, a_n is given by (see Section 10.5)

$$a_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx. \tag{16}$$

Likewise, condition (4) will be satisfied provided that

$$\sum_{n=1}^{\infty} \left(\frac{n\pi c}{l} b_n\right) \sin \frac{n\pi x}{l} = g(x).$$

In other words, b_n is given by

$$b_n = \frac{2}{n\pi c} \int_0^l g(x) \sin \frac{n\pi x}{l}\, dx. \tag{17}$$

We conclude that the solution of the initial-boundary value problem (2)–(6) is given by

$$u(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi c}{l} t + b_n \sin \frac{n\pi c}{l} t\right) \sin \frac{n\pi x}{l}, \tag{18}$$

where a_n and b_n are given by Eqs. (16) and (17) respectively.

EXAMPLE 1 Solve the following initial-boundary value problem.

$$\begin{aligned}
u_{tt} - 4u_{xx} &= 0, && 0 < x < \pi, \quad t > 0, \\
u(x, 0) &= x(\pi - x), && 0 < x < \pi, \\
u_t(x, 0) &= 0, && 0 < x < \pi, \\
u(0, t) &= 0, && t \geq 0, \\
u(\pi, t) &= 0, && t \geq 0.
\end{aligned}$$

Solution The above problem is the initial-boundary value problem (2)–(6) with $c = 2$, $l = \pi$, $f(x) = x(\pi - x)$, $g(x) = 0$. From formula (17) we have $b_n = 0$; applying formula (16) we have (using 49 and 51 in the integral tables)

$$a_n = \frac{2}{\pi}\int_0^\pi x(\pi - x)\sin nx\,dx = \frac{2}{\pi}\left[\pi\int_0^\pi x\sin nx\,dx - \int_0^\pi x^2\sin nx\,dx\right]$$

$$= \frac{2}{\pi}\left\{\pi\left[-\frac{x}{n}\cos nx\,\Big|_0^\pi + \frac{1}{n}\int_0^\pi \cos nx\,dx\right] - \left[-\frac{x^2}{n}\cos nx\,\Big|_0^\pi\right.\right.$$

$$\left.\left. + \frac{2}{n}\int_0^\pi x\cos nx\,dx\right]\right\}$$

$$= \frac{2}{\pi}\left\{\pi\left[\frac{(-1)^{n+1}\pi}{n} + \frac{1}{n^2}\sin nx\,\Big|_0^\pi\right] - \left[\frac{(-1)^{n+1}\pi^2}{n} + \right.\right.$$

$$\left.\left.\frac{2}{\pi}\left(\frac{x}{n}\sin nx\,\Big|_0^\pi - \frac{1}{n}\int_0^\pi \sin nx\,dx\right)\right]\right\}$$

$$= \frac{2}{\pi}\left\{\frac{(-1)^{n+1}\pi^2}{n} - \frac{(-1)^{n+1}\pi^2}{n} + \frac{2}{n^2}\left[-\frac{\cos nx}{n}\,\Big|_0^\pi\right]\right\}$$

$$= \frac{2}{\pi}\left\{\frac{2}{n^2}\left[\frac{1 - (-1)^n}{n}\right]\right\} = \frac{4}{\pi n^3}[1 - (-1)^n].$$

Thus, the solution given by Eq. (18) is

$$u(x, t) = \sum_{n=1}^{\infty} \frac{4}{\pi n^3}[1 - (-1)^n]\cos 2nt\sin nx$$

or

$$u(x, t) = \sum_{n=0}^{\infty} \frac{8}{\pi(2n+1)^3}\cos 2(2n+1)t\sin(2n+1)x.$$

EXAMPLE 2 Solve the following initial-boundary value problem.

$$\begin{aligned}
u_{tt} - 25u_{xx} &= 0, && 0 < x < \pi, \quad t > 0,\\
u(x, 0) &= \sin 3x, && 0 < x < \pi,\\
u_t(x, 0) &= 4, && 0 < x < \pi,\\
u(0, t) &= 0, && t \ge 0,\\
u(\pi, t) &= 0, && t \ge 0.
\end{aligned}$$

Solution Applying formula (16) we have

$$a_n = \frac{2}{\pi}\int_0^\pi \sin 3x\sin nx\,dx.$$

Using the orthogonality property of the functions $\sin nx$, we obtain

$$a_n = \begin{cases} 0, & n \neq 3 \\ \dfrac{2}{\pi}\left[\dfrac{x}{2} - \dfrac{\sin 2(3)x}{4(3)}\right]_0^\pi = 1, & n = 3. \end{cases}$$

From formula (17) we have

$$b_n = \frac{2}{5\pi n}\int_0^\pi 4\sin nx\,dx = \frac{8}{5\pi n}\left[\frac{-\cos nx}{n}\bigg|_0^\pi\right] = \frac{8}{5\pi n^2}[1 - (-1)^n].$$

Thus, the solution of the initial-boundary value problem is

$$\begin{aligned} u(x, t) &= \sum_{n=1}^{\infty} [a_n \cos 5nt + b_n \sin 5nt] \sin nx, \\ &= \cos 15t \sin 3x + \sum_{n=1}^{\infty} \frac{8}{5\pi n^2}[1 - (-1)^n] \sin 5nt \sin nx, \\ &= \cos 15t \sin 3x + \sum_{n=0}^{\infty} \frac{16}{5(2n+1)^2\pi} \sin 5(2n+1)t \sin(2n+1)x. \end{aligned}$$

The method introduced in this section is applicable to many problems that come under the classification of Problem 1, Section 11.5. Other types of equations are presented in Sections 11.7, 11.8, 11.9, and 11.10. We conclude this section with another example involving the wave equation but with different boundary conditions.

EXAMPLE 3 Solve the initial-boundary value problem

$$u_{tt} - c^2 u_{xx} = 0, \qquad 0 < x < l, \quad t > 0 \tag{19}$$

$$u(x, 0) = f(x), \quad 0 < x < l, \tag{20}$$

$$u_t(x, 0) = g(x), \quad 0 < x < l, \tag{21}$$

$$u_x(0, t) = 0, \qquad t > 0, \tag{22}$$

$$u_x(l, t) = 0, \qquad t > 0. \tag{23}$$

Solution Conditions (22) and (23) are different than those imposed for the problem (2)–(6); therefore, if we assume a solution in the form $u(x, t) = X(x)T(t)$, X must satisfy the eigenvalue problem

$$X'' - \frac{\lambda}{c^2} X = 0,$$

$$X'(0) = X'(l) = 0.$$

Except for a slight change in symbolism, this eigenvalue problem is that of Example 2, Section 6.2. Therefore, we have

$$\lambda_n = -\frac{n^2\pi^2c^2}{l^2}, \qquad n = 1, 2, 3, \ldots$$

and

$$X_n(x) = \cos\frac{n\pi x}{l}, \qquad n = 1, 2, 3, \ldots.$$

Thus we can repeat the development of the solution as we did in the beginning of this section. The solution is

$$u(x, t) = \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi c}{l}t + b_n \sin\frac{n\pi c}{l}t\right)\cos\frac{n\pi x}{l},$$

where

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx \tag{24}$$

and

$$b_n = \frac{2}{n\pi c}\int_0^l g(x)\cos\frac{n\pi x}{l}\,dx. \tag{25}$$

That is, a_n and $\left(\frac{n\pi c}{l}\right) b_n$ are the coefficients of the Fourier cosine series of the functions f and g respectively.

EXERCISES

In Exercises 1 through 14, solve the initial-boundary value problem (2)–(6) for the conditions given.

2. $c = 1, l = 1, f(x) = x(1 - x), g(x) = 0$
3. $c = 1, l = \pi, f(x) = x^2(\pi - x), g(x) = 0$
4. $c = 1, l = \pi, f(x) = x(\pi - x)^2, g(x) = 0$
5. $c = 1, l = \pi, f(x) = 0, g(x) = 3$
6. $c = 1, l = \pi, f(x) = 0, g(x) = \pi$
7. $c = 1, l = 1, f(x) = 0, g(x) = A$, A a constant
8. $c = 1, l = \pi, f(x) = 0, g(x) = A$, A a constant
9. $c = 1, l = \pi, f(x) = x(\pi - x), g(x) = 3$
10. $c = 1, l = \pi, f(x) = x^2(\pi - x), g(x) = 3$
11. $c = 1, l = \pi, f(x) = x(\pi - x)^2, g(x) = \pi$
12. $c = 2, l = \pi, f(x) = \sin x, g(x) = \cos x$

13. $c = 4$, $l = \pi$, $f(x) = \sin 5x$, $g(x) = \cos 2x$

14. $c = 3$, $l = \pi$, $f(x) = \sin x$, $g(x) = 0$

In Exercises 15 through 28, solve the initial-boundary value problem (19)–(23) for the conditions given.

15. The same as Exercise 1.

16. The same as Exercise 2.

17. The same as Exercise 3.

18. The same as Exercise 4.

19. The same as Exercise 5.

20. The same as Exercise 6.

21. The same as Exercise 7.

22. The same as Exercise 8.

23. The same as Exercise 9.

24. The same as Exercise 10.

25. The same as Exercise 11.

26. The same as Exercise 12.

27. The same as Exercise 13.

28. The same as Exercise 14.

In Exercises 29 and 30, solve the initial-boundary value problem given $0 \le x \le l$, $t \ge 0$.

29.
$$\begin{aligned} &u_{tt} - c^2u_{xx} = 0 \\ &u(x, 0) = f(x) \\ &u_t(x, 0) = g(x) \\ &u(0, t) = 0 \\ &u_x(l, t) = 0. \end{aligned}$$

30.
$$\begin{aligned} &u_{tt} - c^2u_{xx} = 0 \\ &u(x, 0) = f(x) \\ &u_t(x, 0) = g(x) \\ &u_x(0, t) = 0 \\ &u(l, t) = 0. \end{aligned}$$

31. Suppose that u is a solution of the initial-boundary value problem

$$u_{tt} - c^2u_{xx} = 0, \qquad 0<x<l, \quad t>0, \tag{26}$$

$$u(x, 0) = f(x), \quad 0<x<l, \tag{27}$$

$$u_t(x, 0) = g(x), \quad 0<x<l, \tag{28}$$

$$u(0, t) = A, \qquad t>0, \tag{29}$$

$$u(l, t) = B, \qquad t>0, \tag{30}$$

where A and B are constants. Show that if

$$v(x, t) = u(x, t) + \left(\frac{x-l}{l}\right)A - \frac{x}{l}B, \tag{31}$$

then v is a solution of the initial-boundary value problem

$$\begin{aligned} &v_{tt} - c^2v_{xx} = 0, && 0<x<l, \quad t>0, \\ &v(x, 0) = f(x) + \left(\frac{x-l}{l}\right)A - \frac{x}{l}B, && 0<x<l, \\ &v_t(x, 0) = g(x), && 0<x<l, \\ &v(0, t) = 0, && t>0, \\ &v(l, t) = 0, && t>0. \end{aligned}$$

In Exercises 32 through 45, solve the initial-boundary value problem (26)–(30) of Exercise 31 for the conditions given. [*Hint:* Find v, then determine u from Eq. (31).]

32. $A = 3$, $B = 0$ and the conditions of Exercise 2.

33. $A = 0$, $B = 3$ and the conditions of Exercise 1.

34. $A = -3$, $B = 2$ and the conditions of Exercise 4.

35. $A = 2$, $B = 2$ and the conditions of Exercise 3.

36. $A = 0$, $B = -2$ and the conditions of Exercise 6.

37. $A = 10$, $B = \pi$ and the conditions of Exercise 5.

38. $A = 8\pi$, $B = \pi$ and the conditions of Exercise 8.

39. $A = 7$, $B = 2$ and the conditions of Exercise 7.

40. $A = 4$, $B = 0$ and the conditions of Exercise 10.

41. $A = 0$, $B = 5$ and the conditions of Exercise 9.

42. $A = 9$, $B = 5$ and the conditions of Exercise 12.

43. $A = 0$, $B = 8$ and the conditions of Exercise 11.

44. $A = 13$, $B = -3$ and the conditions of Exercise 14.

45. $A = 6$, $B = 0$ and the conditions of Exercise 13.

46. Give a physical interpretation of the initial conditions of Example 1 (assume that the problem relates to a "string").

47. Give a physical interpretation to the initial conditions of Example 2 (consider that the problem relates to a "string").

48. A tightly stretched string 3 feet long weighs 0.9 lb. and is under a constant tension of 10 lb. The string is initially straight and is set into motion by imparting to each of its points an initial velocity of 1 ft/sec. (a) Find the displacement u as a function of x and t. (b) Find an expression for the displacement of the midpoint one minute after the motion has begun.

49. Wave Equation with Damping If there is a damping force present, for example, air resistance, the equation of the vibrating string becomes

$$u_{tt} + 2\alpha u_t - c^2 u_{xx} = 0, \tag{32}$$

where α is a positive constant known as the *damping factor*.
(a) Set $u(x, t) = e^{-\alpha t}v(x, t)$ and show that v satisfies

$$v_{tt} - \alpha^2 v - c^2 v_{xx} = 0. \tag{33}$$

(b) Set $v(x, t) = w(x)e^{i\beta t}$ in Eq. (33) and show that w satisfies $w'' + \gamma w = 0$, where $\gamma = (\alpha^2 + \beta^2)/c^2$.
(c) Set $v(x, t) = w(x)e^{-i\beta t}$ in Eq. (33) and show that w satisfies $w'' + \gamma w = 0$, where $\gamma = (\alpha^2 + \beta^2)/c^2$.

(d) Utilizing parts (b) and (c), verify that Eq. (32) possesses particular solutions u_1, u_2 of the form

$$u_1 = Ae^{-\alpha t}e^{i(\sqrt{\gamma}x - \beta t)}, \quad u_2 = Be^{-\alpha t}e^{-i(\sqrt{\gamma}x - \beta t)}.$$

(e) Describc u_1, u_2 as "waves" and determine their speed (see Exercise 61b).

50. A taut string of length 1m and $c = 1$ is subjected to air resistance damping for which $\alpha = 1$ (see Exercise 49). Using the method of separation of variables and Fourier series, find the displacement as a function of t and x if the initial displacement is zero and the initial velocity is 1m/sec.

51. Torsional Vibration in Shafts A shaft (rod) of circular cross section has its axis along the x-axis, the ends coinciding with $x = 0$ and $x = l$. The shaft is subjected to a twisting action and then released. $\theta(x, t)$ denotes the *angular* displacement undergone by the mass in the circular cross section located at the position x and time t (that is, the mass of a very thin disc located there). It can be shown that θ satisfies the one-dimensional wave equation with $c = G/\rho$, where G (a constant) is known as the *shear modulus* of the shaft and ρ (a constant) is the density of the shaft. Furthermore, it is known in the theory of elasticity that $\theta_x = \tau/G\mu$, where τ is the *twisting moment* (torque) and μ is the *polar moment* of inertia. Two types of end conditions are common. They are a *fixed end* (for which $\theta = 0$) and a *free end* (for which $\theta_x = 0$, since $\tau = 0$).

Set up and solve the initial-boundary value problem for the angular displacement of a rod that is fixed at the end $x = 0$, free at the end $x = l$ (take $l = 1$), whose initial velocity (θ_t) is zero, and whose initial displacement is given by $3x$. Take $c = 1$.

52. Repeat Exercise 51 by considering all the information to be the same except that the end at $x = 0$ is free instead of fixed, and the end at $x = 1$ is fixed instead of free.

53. Plucked String When the initial displacement of the string is of the form

$$f(x) = \begin{cases} mx, & 0 \le x \le x_0 \\ \dfrac{mx_0}{l - x_0}(l - x), & x_0 \le x \le l \end{cases},$$

one can say that the string has been "pinched" at the point $x = x_0$, lifted (in case $m > 0$) to the height mx_0, and then released. This action is described as *plucking* the string. Find the displacement of the string that is plucked at its midpoint and released from rest. Take $c = l = 1$, $m = \frac{1}{4}$.

Remark It is reasonable to think of a guitar string as being plucked, since a guitar pick or a person's fingernail can be thought of as acting at a point on the string. The same would bc true for a harp string except that the plucking very often occurs at two or more positions on the string. On the other hand, piano strings are set into motion when struck by a hammer, which does not act at a certain point on the string, but rather on a segment of the string. In this case, it is reasonable to think of the string as being in an initial horizontal position; the hammer imparts an initial velocity to the portion of the string it strikes, and the rest of the string has zero initial velocity.

54. **Piano String** Find the displacement of a 2-foot piano string that is struck by a 2-inch hammer having an initial velocity of 1 ft/sec, if it is known that the center of the hammer strikes the center of the string. Take $c = 1$. [*Hint:* See the preceding remark.]

55. A 2-inch-wide acorn travelling at 3 in/sec. strikes a taut spider's web consisting of a single horizontal thread 6 inches long. Find the displacement of the web if it is known that the center of the acorn strikes the web at a point 2 inches from its left end. Take $c = 1$. [*Hint:* See the preceding remark.]

56. **Harmonics** The solution given in Eq. (18) can be written in the form

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t),$$

where

$$u_n(x, t) = A_n g(t)h(x)$$

and

$$A_n = (a_n^2 + b_n^2)^{1/2}, \quad h(x) = \sin\frac{n\pi x}{l}, \quad \text{and} \quad g(t) = \cos\left[\frac{n\pi c}{l}t - \beta\right].$$

(Recall from trigonometry that $A\cos\alpha t + B\sin\alpha t = C\cos(\alpha t - \beta)$, with $C = (A^2 + B^2)^{1/2}$ and $\cos\beta = A/C$.) By itself, u_n is a possible motion of the string and is called the nth *normal* (or *natural*) *mode of vibration* or the nth *harmonic;* u_1 is called the *fundamental mode* or the *fundamental harmonic.* If we consider x to be fixed, then u_n is a simple harmonic motion of period

$$p_n = \frac{2l}{nc}$$

and frequency

$$\nu_n = \frac{nc}{2l}.$$

ν_n is called the nth *natural frequency,* and ν_1 is called the *fundamental frequency* of vibration. If $u_n \equiv 0$, then ν_n is taken to be zero. Find the first four harmonics and the first four natural frequencies for the string of Example 1 of this section.

57. A clothesline 10 feet in length is to be considered as a taut flexible string. A boy strikes the clothesline with a paddle that is one foot wide, so that the paddle is travelling with speed 3 ft/sec at the time of contact. The point of contact of the center of the paddle is at the point 3.5 feet from the left end of the clothesline. Find the displacement of the clothesline. Take $c = 1$.

58. The transverse displacement, $u(x, t)$, of a uniform beam satisfies the partial differential equation $u_{tt} + c^2 u_{xxxx} = 0$, where $c^2 = EI/\rho S$, with E a constant (*Young's modulus*), I a constant (moment of inertia of the cross section area), ρ a constant (the density), and S a constant (the area of cross section).[5] An end of the beam is said to be *hinged* if $u = 0$ and $u_{xx} = 0$ at that end. Set up and solve the initial-boundary value problem for the transverse displacement of a beam of length l that is hinged at both ends, having initial displacement $f(x)$ and initial velocity zero. [*Hint:* Use the method of separation of variables and Fourier series.]

59. If the beam of Exercise 58 is subjected to axial forces, $F(t)$, applied at its ends, the transverse displacement, $u(x, t)$, satisfies[6] the partial differential equation

$$EIu_{xxxx} - F(t)u_{xx} + \rho S u_{tt} = 0.$$

Assume that the beam is of length l and has hinged ends. This equation does not "separate," and we cannot use the method of this section. Set $u(x, t) = \sum_{n=1}^{\infty} T_n(t) \sin \frac{n\pi x}{l}$.

(a) Show that u satisfies the boundary conditions.
(b) Determine a differential equation that $T_n(t)$ must satisfy if u is to be a solution of the partial differential equation.

60. The Telegraph Equation If $u(x, t)$ and $i(x, t)$ represent the voltage and current, respectively, in a cable where t denotes time and x denotes the position in the cable measured from a fixed initial position, the governing equations[7] are

$$Cu_t + Gu + i_x = 0$$

$$Li_t + Ri + u_x = 0.$$

The constants C, G, L, and R represent electrostatic capacity per unit length, leakage conductance per unit length, inductance per unit length, and resistance per unit length, respectively.

(a) Eliminate i from the above system to show that u satisfies the *telegraph equation*

$$LCu_{tt} + (LG + RC)u_t + RGu = u_{xx}. \tag{34}$$

[*Hint:* Differentiate the first equation with respect to t and the second with respect to x.]

(b) Eliminate u from the above system and show that i satisfies Eq. (34).

[5] S. Timoshenko, *Vibration Problems in Engineering* (Princeton, N.J.: Van Nostrand, 1928), p. 221.
[6] Ibid., p. 374.
[7] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, vol. 2 (New York: Interscience Publishers, 1962), p. 193.

61. Travelling Waves Divide by LC and rewrite the telegraph equation, Eq. (34), in the form

$$u_{tt} + (\alpha + \beta)u_t + \alpha\beta u = c^2 u_{xx},$$

with $c^2 = 1/LC$, $\alpha = G/C$, $\beta = R/L$.

(a) Set $u(x, t) = e^{-1/2(\alpha+\beta)t}v(x, t)$, and show that if u is a solution of the telegraph equation, then v satisfies

$$v_{tt} - \left(\frac{\alpha - \beta}{2}\right)^2 v = c^2 v_{xx}.$$

(b) If $\alpha = \beta$, that is, $GL = RC$, then v satisfies the one-dimensional wave equation; although we do not have initial-boundary conditions, we can still discuss the solution. Using the method of Exercise 40, Section 11.4, show that v has the form

$$v(x, t) = f(x + ct) + g(x - ct).$$

The expression $f(x + ct)$ can be thought of as a "wave"[8] travelling to the left with speed c, and $g(x - ct)$ as a "wave" travelling to the right with speed c. Thus the solution u of part (a) can be described as a voltage (or current) subjected to damping that is travelling in both directions in the cable. Thus, for appropriate properties of the cable ($\alpha = \beta$), signals can be transmitted along the cable in a "relatively" undistorted form yet damped in time.

62. Maxwell's Equation for the Electromagnetic Field Intensity in a Homogeneous Medium The partial differential equation

$$(\underset{\sim}{E})_{tt} = \frac{\delta^2}{\mu\epsilon}\Delta[\underset{\sim}{E}] - \frac{4\pi\sigma}{\epsilon}(\underset{\sim}{E})_t \tag{35}$$

is one of *Maxwell's equations* that occur in electrodynamics.[9] $\underset{\sim}{E}$ is a vector representing the *electromagnetic field intensity*, and σ (*conductivity*), ϵ (*dielectric constant*), and μ (*permeability*) are constants associated with the medium; δ is a constant associated with the conversion of units. The one-dimensional version of Eq. (35) can be written in the form (32) with $c = \delta/\sqrt{\mu\epsilon}$, $\alpha = 2\pi\sigma/\epsilon$. Repeat the procedure outlined in Exercise 49. In this application the factor $e^{-\alpha t}$ is called the *attenuation factor*.

63. Determine the tension in a string of length 100 cm. and density 1.5 gram per meter, so that the fundamental frequency of the string is 256 cycles per second (256 cps is middle C on the musical scale).

[8] See, for example, J. M. Pearson, *A Theory of Waves* (Boston: Allyn and Bacon, 1966), p. 2.
[9] Pearson, *Theory of Waves*, p. 29.

11.7 THE ONE-DIMENSIONAL HEAT EQUATION

In the investigation of the flow of heat in a conducting body, the following three laws have been deduced from experimentation.

LAW 1 Heat will flow from a region of higher temperature to a region of lower temperature.

LAW 2 The amount of heat in the body is proportional to the temperature of the body and to the mass of the body.

LAW 3 Heat flows across an area at a rate proportional to the area and to the temperature gradient (that is, the rate of change of temperature with respect to distance where the distance is taken perpendicular to the area).

We consider a rod of length l and constant cross sectional area A. The rod is assumed to be made of material that conducts heat uniformly. The lateral surface of the rod is insulated so that the streamlines of heat are straight lines perpendicular to the cross sectional area A. The x-axis is taken parallel to and in the same direction as the flow of heat. The point $x = 0$ is at one end of the rod and the point $x = l$ is at the other end. ρ denotes the density of the material, and c (a constant) denotes the *specific heat* of the material. (Specific heat is the amount of heat energy required to raise a unit mass of the material one unit of temperature change.) $u(x, t)$ denotes the temperature at time t (> 0) in a cross sectional area A, x units from the end $x = 0$. Consider a small portion of the rod of thickness Δx that is between x and $x + \Delta x$. The amount of heat in this portion is, by Law 2, $c\rho A \Delta x u$. Thus, the time rate of change of this quantity of heat is $c\rho A \Delta x \dfrac{\partial u}{\partial t}$. Thus,

$$c\rho A \Delta x \frac{\partial u}{\partial t} = (\text{rate into this portion}) - (\text{rate out of this portion}).$$

From Law 3 we have

$$\text{rate in} = -kA \left.\frac{\partial u}{\partial x}\right|_{x},$$

$$\text{rate out} = -kA \left.\frac{\partial u}{\partial x}\right|_{x + \Delta x},$$

where the minus sign is a consequence of Law (1) and our assumption regarding the orientation of the x-axis. The constant of proportionality k is known as the *thermal conductivity*. Thus,

$$c\rho A \Delta x \frac{\partial u}{\partial t} = -kA \left.\frac{\partial u}{\partial x}\right|_{x} + kA \left.\frac{\partial u}{\partial x}\right|_{x + \Delta x}$$

or

$$\frac{\partial u}{\partial t} = \frac{k}{c\rho}\left[\frac{\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_{x}}{\Delta x}\right].$$

Taking the limit as Δx tends to zero, we obtain the *one-dimensional heat equation*

$$u_t - au_{xx} = 0, \tag{1}$$

where $a = k/c\rho$ is known as the *diffusivity*. Note that the heat equation is a *parabolic* partial differential equation.

EXAMPLE 1 Solve[10] the initial-boundary value problem

$$u_t - au_{xx} = 0, \qquad 0<x<l, \quad t>0, \tag{2}$$

$$u(0, t) = 0, \qquad t>0, \tag{3}$$

$$u(l, t) = 0, \qquad t>0, \tag{4}$$

$$u(x, 0) = f(x), \quad 0<x<l. \tag{5}$$

Solution We note that conditions (3) and (4) indicate that the ends of the rod are in contact with a heat reservoir of constant temperature zero; (5) gives the initial distribution of temperature. We seek a solution in the form $u(x, t) = X(x)T(t)$. Using the method of separation of variables, we find that X is to be a solution of the eigenvalue problem

$$X'' - \frac{\lambda}{a}X = 0, \tag{6}$$

$$X(0) = 0, X(l) = 0, \tag{7}$$

and T is to satisfy the differential equation

$$T' - \lambda T = 0. \tag{8}$$

The eigenvalues and eigenfunctions of problem (6)–(7) are, respectively,

$$\lambda_n = -\frac{n^2\pi^2 a}{l^2}, \quad n = 1, 2, 3, \ldots, \tag{9}$$

$$X_n(x) = \sin\frac{n\pi x}{l}, \quad n = 1, 2, 3, \ldots. \tag{10}$$

For λ as given in (9), the general solution of Eq. (8) is

$$T_n(t) = c_n e^{-[(n^2\pi^2 a)/l^2]t}.$$

Thus,

$$u(x, t) = \sum_{n=1}^{\infty} X_n(x)T_n(t)$$

[10] It can be shown that if f satisfies the Dirichlet conditions (Theorem 1, Section 10.4), problem (2)–(5) has a unique solution.

is a solution of Eq. (2) that satisfies conditions (3) and (4) and will satisfy condition (5) if

$$\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} = f(x).$$

Thus, the constants c_n are the Fourier sine coefficients of f; that is,

$$c_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx. \tag{11}$$

The solution of the initial-boundary value problem (2)–(5) is

$$u(x, t) = \sum_{n=1}^{\infty} c_n e^{-[(n^2\pi^2 a)/l^2]t} \sin\frac{n\pi x}{l},$$

where c_n is given by Eq. (11).

EXAMPLE 2 If the ends of the rod are in contact with heat reservoirs of constant temperatures A at $x = 0$ and B at $x = l$, the corresponding initial-boundary value problem is

$$u_t - au_{xx} = 0, \qquad 0 < x < l, \quad t > 0, \tag{12}$$

$$u(0, t) = A, \qquad t > 0, \tag{13}$$

$$u(l, t) = B, \qquad t > 0, \tag{14}$$

$$u(x, 0) = f(x), \quad 0 < x < l. \tag{15}$$

Solve the initial-boundary value problem (12)–(15).

Solution If u is a solution of the initial-boundary value problem (12)–(15), then

$$v(x, t) = u(x, t) + \left(\frac{x - l}{l}\right)A - \frac{x}{l}B \tag{16}$$

is a solution of the initial-boundary value problem

$$\begin{aligned} v_t - av_{xx} &= 0, && 0 < x < l, \quad t > 0, \\ v(0, t) &= 0, && t > 0, \\ v(l, t) &= 0, && t > 0, \\ v(x, 0) &= f(x) + \left(\frac{x - l}{l}\right)A - \frac{x}{l}B, && 0 < x < l. \end{aligned}$$

Thus the solution is obtained by first finding v using the method of Example 1, and then determining u from Eq. (16).

EXERCISES

In Exercises 1 though 16, solve the initial-boundary value problem (2)–(5) for the given conditions.

1. $a = 1, l = 1, f(x) = x$

2. $a = 1, l = \pi, f(x) = x$

3. $a = 4, l = \pi, f(x) = x^2$

4. $a = 4, l = \pi, f(x) = \sin x$

5. $a = 2, l = 1, f(x) = \cos \dot{x} + 3x$

6. $a = 1, l = 3, f(x) = x + 3$

7. $a = 5, l = 1, f(x) = e^x$

8. $a = 3, l = \pi, f(x) = \sin 3x$

9. $a = 1, l = \pi, f(x) = x + \sin x$

10. $a = 2, l = 1, f(x) = \sin^2 x$

11. $a = 1, l = 1, f(x) = x(1 - x)$

12. $a = 1, l = 1, f(x) = x^2(1 - x)$

13. $a = 2, l = \pi, f(x) = x(\pi - x)^2$

14. $a = 1, l = 1, f(x) = x^3$

15. $a = 1, l = 1, f(x) = x + e^x$

16. $a = 1, l = 1, f(x) = x^2 + e^x$

In Exercises 17 through 25, solve the initial-boundary value problem (12)–(15) for the given conditions.

17. $a = 5, l = \pi, f(x) = x, A = 10, B = 0$

18. $a = 3, l = \pi, f(x) = x, A = 0, B = -5$

19. $a = 1, l = 1, f(x) = x(1 - x), A = 7, B = 3$

20. $a = 2, l = 1, f(x) = x^2, A = 5, B = 5$

21. $a = 1, l = 1, f(x) = \sin^2 x, A = 7, B = 2$

22. $a = 1, l = 1, f(x) = x + 3, A = 6, B = 4$

23. $a = 3, l = 1, f(x) = e^x, A = 7, B = 3$

24. $a = 1, l = 1, f(x) = x + e^x, A = 25, B = 15$

25. $a = 5, l = 1, f(x) = x(1 - x)^2, A = 10, B = 10$

Remark In general the heat equation can be written in the form

$$u_t = a\Delta[u], \tag{17}$$

where Δ is the Laplacian operator introduced in Exercise 50, Section 11.3.

Consequently one can consider heat diffusion problems in two and three dimensions also. Some of these higher-dimensional problems can be handled in precisely the same manner as in this section; however, the analysis is complicated by the fact that the corresponding eigenvalue problems involve Bessel and Legendre functions. These functions are considerably more difficult to deal with than the sine and cosine functions heretofore encountered. For this reason we have restricted our attention to the one-dimensional heat equation. There are, however, some special types of two- and three-dimensional problems that lead to relatively simple solutions (see Exercises 26, and 32 through 36).

26. If the temperature is a function of the radius only, show that the two-dimensional heat equation can be written in the form

$$u_t = \frac{a}{r}(ru_r)_r. \tag{18}$$

[*Hint:* See Exercise 50, Section 11.3.]

27. Heat Equation: Source Terms If internal sources of heat are present in the rod and if the rate of production of heat is the same throughout any given cross section of area, then the equation[11] of heat flow is

$$u_t - au_{xx} = q(x, t, u), \tag{19}$$

where $h(x, t, u) = c\rho q(x, t, u)$ is the rate of production of heat energy per unit volume per unit time. The following special cases are of interest.

(a) $q(x, t, u) = F(x, t)$. This situation corresponds to heating caused by an electric current through the rod and gives rise to a nonhomogeneous problem. See Exercise 24, Section 11.9 for a specific example.

(b) $q(x, t, u) = r(x, t)u$. This case can be thought of as heating of the rod caused by a chemical reaction that is proportional to the local temperature. If r is positive, the rod is receiving heat (thus, a heat *source* exists); if r is negative, the rod is giving off heat (thus, a heat *loss* exists).

Set up and solve the initial-boundary value problem consisting of Eq. (19) with $q = -\alpha u$, $\alpha(> 0)$ a constant, $u(x, 0) = x(1 - x)$, $u(0, t) = u(l, t) = 0$, $l = a = 1$. [*Hint:* Set $u = e^{-\alpha t} v(x, t)$; the initial-boundary value problem for v is solvable by the methods of this section.]

28. Automatic Heat Control The initial-boundary value problem

$$\begin{aligned} u_t &= au_{xx}, && 0 < x < l, \quad t > 0 \\ u_x(0, t) &= 0, && t > 0, \\ u(l, t) &= Au(0, t), && t > 0, \\ u(x, 0) &= f(x), && 0 < x < l, \end{aligned}$$

[11] P. W. Berg and J. L. McGregor, *Elementary Partial Differential Equations* (San Francisco: Holden-Day, 1966), p. 32.

where A is a constant ($A \neq 1$), describes the temperature in a rod whose left end ($x = 0$) is thermally insulated (that is, no heat is lost through this end, hence $u_x = 0$), and whose right end is equipped with an *automatic heat control* which keeps the temperature at this end proportional to the temperature at the left end. The initial distribution of temperature in the rod is $f(x)$.
(a) If $-1 < A < 1$, solve this initial-boundary value problem.
(b) Show that there are no real eigenvalues in the case $A < -1$.

29. Solve the initial-boundary value problem (2)–(5) when

$$f(x) = \begin{cases} A, & 0 < x < \dfrac{l}{2}, \\ 0, & \dfrac{l}{2} < x < l, \end{cases}$$

where A is a constant.

30. Two rods of the same type of material, each 10 cm. long, are placed face to face in perfect contact. The outer faces are maintained at 0°C. Just prior to contact, one of the rods has constant temperature 100°C; the other has constant temperature 0°C throughout. (a) Find the temperature distribution as a function of x and t after contact is made. [*Hint:* See Exercise 29.] (b) If $a = 0.2$ for the material, find to the nearest degree the temperature at a point on the common face and at points 5 cm. from this common face twenty minutes after contact is made.

31. Three rods of the same material, $a = 1.02$, are each 10 cm. long. The rods each have constant temperature, two at 0°C and one at 100°C. The rods are placed end to end, with the hot rod in the middle. Find the temperature at a point on the middle (center) cross section of the "new" rod if the ends of the "new" rod are maintained at 0°C.

32. (a) Referring to Exercise 50(c) of Section 11.3 and the Remark which follows that exercise, show that the heat equation in spherical coordinates for a temperature distribution that depends only on the radius r and the time t is

$$u_t = \frac{a}{r^2}(r^2 u_r)_r.$$

(b) Show that $\dfrac{1}{r^2}(r^2 u_r)_r = \dfrac{1}{r}(ru)_{rr}$.

33. Temperature Distribution in a Sphere The surface of a homogeneous solid sphere of radius R is maintained at the constant temperature 0°C and has an initial temperature distribution given by $g(r)$. If $u(r, t)$ denotes the temperature in the sphere as a function of the radius r and the time t only, then $u(r, t)$ satisfies the problem (see Exercise 32)

$$\begin{aligned} u_t &= \frac{a}{r}(ru)_{rr}, && 0 < r < R, \quad t > 0 \\ u(R, t) &= 0, && t > 0, \\ u(r, 0) &= g(r), && 0 \le r < R. \end{aligned}$$

It is reasonable to assume that u is bounded at $r = 0$; therefore, if we set $v(r, t) = ru(r, t)$, we would have $v(0, t) = 0$. (a) Set up and solve the corresponding initial-boundary value problem for v. (b) Find u.

34. Repeat Exercise 33 for $g(r) = A$, a constant.

35. Repeat Exercise 33 for $g(r) = R^2 - r^2$.

36. A solid sphere of radius 10 cm. is made of material for which $a = 0.2$. Initially the temperature is 100°C throughout the sphere, and the sphere is cooled by keeping the surface of the sphere at 0°C. Find the temperature (to the nearest degree) at the center of the sphere 20 minutes after the cooling begins. [*Hint:* See Exercise 33.]

11.8 THE POTENTIAL (LAPLACE) EQUATION

We begin this section with a consideration of heat flow in two dimensions. Refer to Section 11.7 for the appropriate definitions and the experimental laws associated with heat flow.

The conducting material is in the shape of a rectangular sheet of constant thickness D, and consider the flow of heat in the differential portion of the sheet depicted in Figure 11.2. Using the same arguments as in Section 11.7, we determine that the gain of heat in the horizontal direction for this portion is

$$kD\Delta y \left[\frac{\partial u}{\partial x}\bigg|_{x + \Delta x} - \frac{\partial u}{\partial x}\bigg|_{x} \right],$$

and in the vertical direction it is

$$kD\Delta x \left[\frac{\partial u}{\partial y}\bigg|_{y + \Delta y} - \frac{\partial u}{\partial y}\bigg|_{y} \right].$$

Thus, the total gain of heat for this portion of the sheet is

$$kD\left\{ \Delta y \left[\frac{\partial u}{\partial x}\bigg|_{x + \Delta x} - \frac{\partial u}{\partial x}\bigg|_{x} \right] + \Delta x \left[\frac{\partial u}{\partial y}\bigg|_{y + \Delta y} - \frac{\partial u}{\partial y}\bigg|_{y} \right] \right\},$$

or

$$kD\Delta x\Delta y \left[\frac{\dfrac{\partial u}{\partial x}\bigg|_{x + \Delta x} - \dfrac{\partial u}{\partial x}\bigg|_{x}}{\Delta x} + \frac{\dfrac{\partial u}{\partial y}\bigg|_{y + \Delta y} - \dfrac{\partial u}{\partial y}\bigg|_{y}}{\Delta y} \right].$$

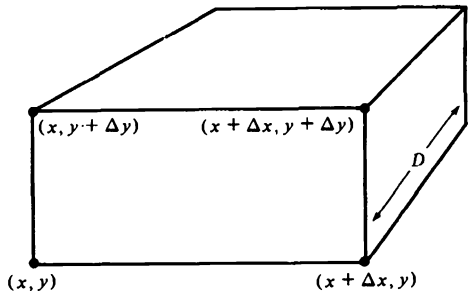


Figure 11.2

By Law 2 of Section 11.7, this rate of gain of heat is also given by $c\rho D\Delta x\Delta y\dfrac{\partial u}{\partial t}$. Hence,

$$c\rho D\Delta x\Delta y\frac{\partial u}{\partial t} = kD\Delta x\Delta y\left[\frac{\left.\dfrac{\partial u}{\partial x}\right|_{x+\Delta x} - \left.\dfrac{\partial u}{\partial x}\right|_{x}}{\Delta x} + \frac{\left.\dfrac{\partial u}{\partial y}\right|_{y+\Delta y} - \left.\dfrac{\partial u}{\partial y}\right|_{y}}{\Delta y}\right].$$

Simplifying and taking the limit as both Δx and Δy tend to zero, we obtain the two-dimensional heat equation

$$u_t = a(u_{xx} + u_{yy}),$$

where $a = k/\rho c$.

If at a future point in time, the temperature u is a function of x and y only (that is, u is independent of time and $u_t = 0$), one says that *steady state* conditions have been achieved. Consequently, the partial differential equation for the steady state temperature for two-dimensional heat flow is

$$u_{xx} + u_{yy} = 0. \qquad (1)$$

Equation (1) is called the *two-dimensional potential (Laplace) equation.* Note that the potential equation is an *elliptic* partial differential equation.

In addition to the steady state temperature for two-dimensional heat flow, there are many other physical applications in which the potential equation occurs. One application is that u represents the potential of a two-dimensional electrostatic field. In this case and for that of two-dimensional heat flow (as well as others), a common situation is that u is to satisfy Eq. (1) in a fixed bounded region, say R, of the xy plane, and that the values of u are known on the boundaries of this region. Thus, a typical problem associated with the potential equation is a boundary value problem, commonly referred to as *Dirichlet's problem,* for R. It is the geometry of R and the nature of the boundary conditions that dictate whether or not it is easy (or even possible) to solve the Dirichlet problem.

EXAMPLE 1 Solve[12] the following Dirichlet problem on the rectangle $0 < x < l$, $0 < y < m$.

$$u_{xx} + u_{yy} = 0, \qquad 0<x<l, \quad 0<y<m, \tag{2}$$

$$u(x, 0) = f(x), \quad 0<x<l, \tag{3}$$

$$u(x, m) = g(x), \quad 0<x<l, \tag{4}$$

$$u(0, y) = 0, \qquad 0<y<m, \tag{5}$$

$$u(l, y) = 0, \qquad 0<y<m. \tag{6}$$

Solution We seek a solution of Eq. (2) of the form $u(x, y) = X(x)Y(y)$. Thus, by the separation of variables method, we find that X must satisfy the eigenvalue problem

$$X'' + \lambda X = 0, \tag{7}$$

$$X(0) = 0, X(l) = 0, \tag{8}$$

and Y must satisfy the differential equation

$$Y'' - \lambda Y = 0. \tag{9}$$

The eigenvalues and eigenfunctions of problem (7)–(8) are given respectively by

$$\lambda_n = \frac{n^2\pi^2}{l^2}, \qquad n = 1, 2, 3, \ldots, \tag{10}$$

$$X_n(x) = \sin\frac{n\pi x}{l}, \quad n = 1, 2, 3, \ldots. \tag{11}$$

For λ as given in (10), the general solution of the differential equation (9) is

$$Y_n(y) = a_n \cosh\frac{n\pi y}{l} + b_n \sinh\frac{n\pi y}{l}. \tag{12}$$

Thus we seek a solution of the boundary value problem (2)–(6) in the form

$$u(x, y) = \sum_{n=1}^{\infty} X_n(x)Y_n(y). \tag{13}$$

Now (13) satisfies each of (2), (5), and (6) and will satisfy (3) if

$$\sum_{n=1}^{\infty} Y_n(0)\sin\frac{n\pi x}{l} = f(x).$$

Also, (13) will satisfy condition (4) if

$$\sum_{n=1}^{\infty} Y_n(m)\sin\frac{n\pi x}{l} = g(x).$$

[12] It can be shown that if f and g satisfy the Dirichlet conditions (Theorem 1, Section 10.4), problem (2)–(6) has a unique solution. Also, for the potential equation, we assume that $l \neq \infty$, $m \neq \infty$.

We conclude that $Y_n(0)$ should be the Fourier sine coefficients of f, and $Y_n(m)$ should be the Fourier sine coefficients of g; in other words,

$$Y_n(0) = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx,$$

and

$$Y_n(m) = \frac{2}{l}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx.$$

Now

$$Y_n(0) = a_n$$

and

$$Y_n(m) = a_n\cosh\frac{n\pi m}{l} + b_n\sinh\frac{n\pi m}{l}.$$

Hence,

$$a_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx \tag{14}$$

and

$$b_n = \frac{1}{\sinh\frac{n\pi m}{l}}\left[\frac{2}{l}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx - \left(\cosh\frac{n\pi m}{l}\right)\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx\right]. \tag{15}$$

We conclude that the solution of the boundary value problem (2)–(6) is given by

$$u(x, y) = \sum_{n=1}^{\infty}\left[a_n\cosh\frac{n\pi y}{l} + b_n\sinh\frac{n\pi y}{l}\right]\sin\frac{n\pi x}{l}, \tag{16}$$

with a_n and b_n given by Eqs. (14) and (15) respectively.

EXAMPLE 2 Solve the following Dirichlet problem for the rectangle $0 < x < l,\ \ 0 < y < m$.

$$u_{xx} + u_{yy} = 0, \qquad 0<x<l, \quad 0<y<m, \tag{17}$$

$$u(x, 0) = 0, \qquad 0<x<l, \tag{18}$$

$$u(x, m) = 0, \qquad 0<x<l, \tag{19}$$

$$u(0, y) = h(y), \quad 0<y<m, \tag{20}$$

$$u(l, y) = k(y), \quad 0<y<m. \tag{21}$$

Solution Using the method of separation of variables, we find that Y must satisfy the eigenvalue problem

$$Y'' + \lambda Y = 0,$$
$$Y(0) = 0, \ Y(m) = 0,$$

and X must satisfy the differential equation

$$X'' - \lambda X = 0.$$

Note that the problems we have to solve are analogous to those of Example 1 except that the roles of X and Y have interchanged. Consequently we can borrow the results of Example 1 to conclude that the solution of the boundary value problem (17)–(21) is given by

$$u(x, y) = \sum_{n=1}^{\infty} \left[\alpha_n \cosh \frac{n\pi x}{m} + \beta_n \sinh \frac{n\pi x}{m}\right] \sin \frac{n\pi y}{m}, \tag{22}$$

where

$$\alpha_n = \frac{2}{m} \int_0^m h(y) \sin \frac{n\pi y}{m} \, dy$$

and

$$\beta_n = \frac{1}{\sinh \dfrac{n\pi l}{m}} \left[\frac{2}{m} \int_0^m k(y) \sin \frac{n\pi y}{m} dy - \left(\cosh \frac{n\pi l}{m}\right) \frac{2}{m} \int_0^m h(y) \sin \frac{n\pi y}{m} dy\right].$$

EXAMPLE 3 Solve the following Dirichlet problem for the rectangle $0 < x < l, \ 0 < y < m$.

$$u_{xx} + u_{yy} = 0, \qquad 0 < x < l, \ 0 < y < m, \tag{23}$$
$$u(x, 0) = f(x), \quad 0 < x < l, \tag{24}$$
$$u(x, m) = g(x), \quad 0 < x < l. \tag{25}$$
$$u(0, y) = h(y), \quad 0 < y < m, \tag{26}$$
$$u(l, y) = k(y), \quad 0 < y < m. \tag{27}$$

Solution We consider the two separate problems

$$u_{xx} + u_{yy} = 0, \qquad 0 < x < l, \ 0 < y < m,$$
$$u(x, 0) = f(x), \quad 0 < x < l,$$
$$u(x, m) = g(x), \quad 0 < x < l,$$
$$u(0, y) = 0, \qquad 0 < y < m,$$
$$u(l, y) = 0, \qquad 0 < y < m,$$

and

$$\begin{aligned} u_{xx} + u_{yy} &= 0, && 0 < x < l, \quad 0 < y < m \\ u(x, 0) &= 0, && 0 < x < l, \\ u(x, m) &= 0, && 0 < x < l, \\ u(0, y) &= h(y), && 0 < y < m, \\ u(l, y) &= k(y), && 0 < y < m. \end{aligned}$$

The solution of the first problem will be denoted by $u_1(x, y)$ and is given by formula (16). The solution of the second problem will be denoted by $u_2(x, y)$ and is given by formula (22). Since Eq. (23) is a homogeneous linear partial differential equation, we can use the principle of superposition to conclude that the solution of the boundary value problem (23)–(27) is

$$u(x, y) = u_1(x, y) + u_2(x, y).$$

EXERCISES

In Exercises 1 through 16, certain conditions are given. In each case find the solution of the boundary value problem (2)–(6).

1. $l = 1, m = 1, f(x) = x, g(x) = 0$

2. $l = 1, m = 1, f(x) = 0, g(x) = x$

3. $l = 1, m = 1, f(x) = x^2, g(x) = 0$

4. $l = 1, m = 1, f(x) = 0, g(x) = x^2$

5. $l = 1, m = 1, f(x) = \sin \pi x, g(x) = 0$

6. $l = 1, m = 1, f(x) = 0, g(x) = \sin \pi x$

7. $l = 1, m = 2, f(x) = 0, g(x) = \cos x$

8. $l = 1, m = 2, f(x) = \cos x, g(x) = 0$

9. $l = 2, m = 1, f(x) = 0, g(x) = e^x$

10. $l = 2, m = 1, f(x) = e^x, g(x) = 0$

11. $l = 1, m = 1, f(x) = x, g(x) = \sin \pi x$

12. $l = 1, m = 1, f(x) = \sin \pi x, g(x) = x^2$

13. $l = 2, m = 2, f(x) = e^x, g(x) = \cos x$

14. $l = 2, m = 2, f(x) = \cos x, g(x) = e^x$

15. $l = 1, m = 2, f(x) = x^2, g(x) = e^x$

16. $l = 1, m = 3, f(x) = \sin x, g(x) = x^2$

In Exercises 17 through 32, certain conditions are given. In each case find the solution of the boundary value problem (17)–(21).

17. $l = 1, m = 1, h(y) = \sin y, k(y) = 0$

18. $l = 1, m = 1, h(y) = 0, k(y) = \sin y$

19. $l = 1, m = 1, h(y) = 0, k(y) = y$

20. $l = 1, m = 1, h(y) = y, k(y) = 0$

21. $l = 1, m = 1, h(y) = \cos y, k(y) = 0$

22. $l = 1, m = 1, h(y) = 0, k(y) = \cos y$

23. $l = 1, m = 2, h(y) = 0, k(y) = y^2$

24. $l = 1, m = 2, h(y) = y^2, k(y) = 0$

25. $l = 2, m = 1, h(y) = e^y, k(y) = 0$

26. $l = 2, m = 1, h(y) = 0, k(y) = e^y$

27. $l = 1, m = 1, h(y) = e^{3y}, k(y) = 3y$

28. $l = 1, m = 1, h(y) = \cos y, k(y) = e^y$

29. $l = 2, m = 2, h(y) = y + 2, k(y) = 0$

30. $l = 2, m = 2, h(y) = e^y, k(y) = \cos y$

31. $l = 3, m = 2, h(y) = 1, k(y) = 3$

32. $l = 2, m = 3, h(y) = \sin y, k(y) = 0$

In Exercises 33 through 40, certain conditions are given. In each case solve the boundary value problem (23)–(27).

33. $l = 1, m = 1, f(x) = x, g(x) = 0, h(y) = \sin y, k(y) = 0$

34. $l = 1, m = 1, f(x) = 0, g(x) = x^2, h(y) = 0, k(y) = \cos y$

35. $l = 2, m = 1, f(x) = 0, g(x) = e^x, h(y) = e^y, k(y) = 0$

36. $l = 1, m = 1, f(x) = \sin \pi x, g(x) = 0, h(y) = \cos y, k(y) = 0$

37. $l = 1, m = 1, f(x) = \sin \pi x, g(x) = x^2, h(y) = \sin y, k(y) = 0$

38. $l = 2, m = 1, f(x) = e^x, g(x) = 0, h(y) = 0, k(y) = e^y$

39. $l = 2, m = 2, f(x) = \cos x, g(x) = e^x, h(y) = e^y, k(y) = \cos y$

40. $l = 1, m = 1, f(x) = 0, g(x) = \sin \pi x, h(y) = 0, k(y) = \cos y$

41. A rectangular homogeneous thermally conducting sheet of material (of small thickness) of width 10 cm. and height 15 cm. lies in the first quadrant with one vertex at the origin and one side on the vertical axis. Find the steady state temperature in the sheet if the edge $x = 0$ is maintained at 100°C and the other three edges are maintained at 0°C.

42. Laplace's equation in polar coordinates is (Exercise 50(b), Section 11.3)

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

(a) Set $u = R(r)\Theta(\theta)$ and determine the differential equations for R and Θ.
(b) Solve the differential equations of part (a). [*Hint:* For R see Section 2.7.]

43. **Dirichlet's Problem for a Circular Region** When the region R for Dirichlet's problem is a circle, the following initial-boundary value problem results.

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \qquad 0 < r < a, \quad -\infty < \theta < \infty \tag{28}$$

$$u(a, \theta) = f(\theta), \quad 0 \le \theta \le 2\pi, \tag{29}$$

$$u(r, -\pi) - u(r, \pi) = 0, \qquad 0 \le r \le a, \tag{30}$$

$$u_\theta(r, -\pi) - u_\theta(r, \pi) = 0, \qquad 0 \le r \le a, \tag{31}$$

$$u(r, \theta) \text{ is continuous at } r = 0. \tag{32}$$

f is assumed to be periodic of period 2π. Conditions (30) and (31) indicate that u is periodic as a function of θ, and condition (32) demands that u be continuous throughout the circular region. Solve the boundary value problem (28)–(32). [*Hint:* The solutions from part (b) of Exercise 42 are of the form

$$(c_1 + c_2 \log r)(b_1 + b_2\theta) + (c_3 r^{\sqrt{\lambda}} + c_4 r^{-\sqrt{\lambda}})(b_3 \cos\sqrt{\lambda}\theta + b_4 \sin\sqrt{\lambda}\theta).$$

The periodicity conditions require that $b_2 = 0$ and $\lambda = \lambda_n = n^2$, $n = 0, 1, 2, \ldots$. The continuity condition requires that $c_2 = c_4 = 0$. Thus, if

$$u_n(r, \theta) = r^n(\alpha_n \cos n\theta + \beta_n \sin n\theta),$$

we can write

$$u(r, \theta) = \sum_{n=0}^{\infty} u_n(r, \theta).$$

Complete the solution.]

44. **Dirichlet's Problem for a Semicircular Region** Set up the boundary value problem for the steady state temperature, u, in a semicircular region, if the temperature along the diameter is zero and if the temperature along the semicircle is a known function of θ, say $f(\theta)$. [*Hint:* Consider the center of the semicircle to be at the origin and the semicircle to lie in the upper half plane, so that $0 \le r \le a$ and $0 \le \theta \le \pi$. Note that $u(r, 0) = 0$ and $u(r, \pi) = 0$. See also Exercise 43.]

45. Solve the boundary value problem of Exercise 44 whre $f(\theta) = 100°C$, $a = 1$.

46. Set up the boundary value problem for the steady state temperature in a

circular section $0 \le r \le a$, $0 \le \theta \le \alpha < \pi$ if $u = 0$ on $\theta = 0$, and $\theta = \alpha$ and $u = f(\theta)$ on the circular segment. [*Hint:* See Exercise 44.]

11.9 NONHOMOGENEOUS PARTIAL DIFFERENTIAL EQUATIONS: METHOD I

Our objective is to solve Problem 1 of Section 11.5 with $F \neq 0$. We illustrate the approach with some examples.

EXAMPLE 1 Solve the initial-boundary value problem

$$u_t - au_{xx} = F(x, t), \quad 0 < x < l, \quad t > 0, \tag{1}$$

$$u(0, t) = 0, \qquad t > 0, \tag{2}$$

$$u(l, t) = 0, \qquad t > 0, \tag{3}$$

$$u(x, 0) = f(x), \qquad 0 < x < l. \tag{4}$$

Solution If we set $F(x, t) = 0$ in Eq. (1) and leave the rest of the problem alone, then we have Example 1 of Section 11.7. The solution for this case is

$$\sum_{n=1}^{\infty} c_n e^{-(n^2\pi^2 at)/l^2} \sin \frac{n\pi x}{l},$$

where

$$c_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx.$$

We introduce the notation $X_n(x) = \sin \dfrac{n\pi x}{l}$ and

$$\phi_n(x) = \sqrt{\frac{2}{l}} \sin \frac{n\pi x}{l}. \tag{5}$$

Applying Theorem 1 of Section 6.2, we have

$$(\phi_n, \phi_m) = \begin{cases} 0, & \text{if } n \neq m \\ 1, & \text{if } n = m \end{cases}.$$

If $\psi(x)$ is twice continuously differentiable and satisfies $\psi(0) = \psi(l) = 0$, then $\psi(x) = \sum_{n=1}^{\infty} (\psi, \phi_n)\phi_n$.

We seek a solution of (1)–(4) in the form

$$u(x, t) = \sum_{n=1}^{\infty} T_n(t)\phi_n(x),$$

where $T_n(t)$ are unknown functions to be determined. By the orthonormality of ϕ_n (Theorem 1 of Section 6.2), we can write

$$T_n(t) = (u, \phi_n) = \int_0^l u(x, t)\phi_n(x)dx. \tag{6}$$

Assuming that differentiation under the integral sign is permissible, we have

$$T_n'(t) = \int_0^l u_t(x, t)\phi_n(x)dx.$$

From Eq. (1) we know that $u_t = au_{xx} + F(x, t)$; hence,

$$\begin{aligned} T_n'(t) &= \int_0^l [au_{xx}(x, t) + F(x, t)]\phi_n(x)dx \\ &= a\int_0^l u_{xx}(x, t)\phi_n(x)dx + \int_0^l F(x, t)\phi_n(x)dx. \end{aligned} \tag{7}$$

Since $F(x, t)$ and $\phi_n(x)$ are known, the second integral in Eq. (7) is a known function of t, say $K_n(t)$. For the first integral in Eq. (7) we employ integration by parts twice to obtain

$$\begin{aligned} \int_0^l u_{xx}(x, t)\phi_n(x)dx &= u_x(x, t)\phi_n(x)\Big|_{x=0}^{x=l} - \int_0^l u_x(x, t)\phi_n'(x)dx \\ &= [u_x(x, t)\phi_n(x) - u(x, t)\phi_n'(x)]\Big|_{x=0}^{x=l} + \int_0^l u(x, t)\phi_n''(x)dx. \end{aligned} \tag{8}$$

Since

$$\phi_n(0) = \phi_n(l) = u(0, t) = u(l, t) = 0,$$

the first term disappears and we have

$$\int_0^l u_{xx}(x, t)\phi_n(x)dx = \int_0^l u(x, t)\phi_n''(x)dx.$$

Furthermore, $\phi_n'' + \mu_n\phi_n = 0$ implies that $\phi_n'' = -\mu_n\phi_n$, where $\mu_n = \dfrac{n^2\pi^2}{l^2}$ [see Eqs. (6) and (9), Section 11.7], hence

$$\int_0^l u_{xx}(x, t)\phi_n(x)dx = -\mu_n\int_0^l u(x, t)\phi_n(x)dx = -\mu_n T_n(t).$$

Substituting our results in Eq. (7), we see that $T_n(t)$ must satisfy the ordinary differential equation

$$T_n'(t) + a\mu_n T_n(t) = K_n(t). \tag{9}$$

Equation (9) is a first-order linear ordinary differential equation and can be solved by the method of Section 1.4. Thus,

$$T_n(t) = e^{-a\mu_n t}\left[\int_0^t K_n(s)e^{a\mu_n s}\,ds + c\right]. \tag{10}$$

From Eqs. (5) and (6), we note that

$$T_n(0) = \int_0^l u(x, 0)\phi_n(x)dx = \int_0^l f(x)\phi_n(x)dx.$$

Consequently, the constant c in Eq. (10) is given by

$$c = T_n(0) = \int_0^l f(x)\phi_n(x)dx. \tag{11}$$

Finally, the solution of the initial-boundary value problem (1)–(4) is given by

$$u(x, t) = \sum_{n=1}^{\infty} T_n(t)\phi_n(x),$$

where $\phi_n(x)$ is given by Eq. (5) and $T_n(t)$ is given by Eqs. (10) and (11).

The method of solution outlined in Example 1 is useful for many problems of the type encountered in Sections 11.6, 11.7, and 11.8. It is important that the associated problem obtained by setting $F(x, t) = 0$ is solvable and that the "boundary terms" in the integration by parts process [see Eq. (8)] either vanish or are known as a function of t. The latter requirement will be met if the boundary conditions for the original problem are $u(0, t) = 0$ and $u(l, t) = 0$. If this is not the case, it is sometimes possible to introduce a change of variables that makes the boundary conditions have this form (see Section 11.10).

EXAMPLE 2 Solve the initial-boundary value problem

$$u_{tt} - c^2u_{xx} = F(x, t), \quad 0 < x < l, \quad t > 0 \tag{12}$$

$$u(x, 0) = f(x), \quad 0 < x < l, \tag{13}$$

$$u_t(x, 0) = g(x), \quad 0 < x < l, \tag{14}$$

$$u(0, t) = 0, \quad t > 0, \tag{15}$$

$$u(l, t) = 0, \quad t > 0. \tag{16}$$

Solution If we set $F(x, t) = 0$ in Eq. (12), we then have the initial-boundary value problem of Section 11.6. As in Example 1, we set

$$\phi_n(x) = \sqrt{2/l}\sin\frac{n\pi x}{l},$$

and we seek a solution of problem (12)–(16) in the form

$$u(x, t) = \sum_{n=1}^{\infty} T_n(t)\phi_n(x),$$

where the functions $T_n(t)$ are to be determined. By Theorem 1 of Section 6.2, we can write

$$T_n(t) = (u, \phi_n) = \int_0^l u(x, t)\phi_n(x)dx. \tag{17}$$

Assuming that differentiation under the integral sign is permissible, we have

$$T_n'(t) = \int_0^l u_t(x, t)\phi_n(x)dx, \tag{18}$$

$$T_n''(t) = \int_0^l u_{tt}(x, t)\phi_n(x)dx. \tag{19}$$

Solving Eq. (12) for u_{tt} and substituting the result in Eq. (19) leads to

$$\begin{aligned} T_n''(t) &= \int_0^l [c^2 u_{xx}(x, t) + F(x, t)]\phi_n(x)dx \\ &= c^2 \int_0^l u_{xx}(x, t)\phi_n(x)dx + \int_0^l F(x, t)\phi_n(x)dx. \end{aligned}$$

Setting

$$K_n(t) = \int_0^l F(x, t)\phi_n(x)dx,$$

and utilizing integration by parts, we have

$$\begin{aligned} T_n''(t) &= c^2[u_x(x, t)\phi_n(x) - u(x, t)\phi_n'(x)] \Big|_{x=0}^{x=l} \\ &\quad + c^2 \int_0^l u(x, t)\phi_n''(x)dx + K_n(t). \end{aligned}$$

As in Example 1, the "boundary term" disappears and $\phi_n'' = -\mu_n\phi_n$ with $\mu_n = \dfrac{n^2\pi^2}{l^2}$ [see Eq. (10), Section 11.6]. Thus,

$$T_n'' = c^2 \int_0^l u(x, t)[-\mu_n\phi_n(x)]dx + K_n(t).$$

In other words, T_n must satisfy the ordinary differential equation

$$T_n''(t) + c^2\mu_n T_n(t) = K_n(t). \tag{20}$$

Equation (20) is a second-order nonhomogeneous linear ordinary differential equation and can be solved by the method of Section 2.11 or of Section 2.12, depending on the function $K_n(t)$. Coupling Eqs. (13) and (17) and Eqs. (14) and (18), we find that $T_n(t)$ must satisfy the initial conditions

$$T_n(0) = \int_0^l f(x)\phi_n(x)dx, \tag{21}$$

$$T_n'(0) = \int_0^l g(x)\phi_n(x)dx. \tag{22}$$

If the function $K_n(t)$ is continuous for $t > 0$, the initial value problem (20)–(22) has a unique solution $T_n(t)$, and the solution of the original initial-boundary value problem (12)–(16) is given by

$$u(x, t) = \sum_{n=1}^{\infty} T_n(t)\phi_n(x),$$

where $\phi_n(x)$ is given by Eq. (5) and $T_n(t)$ is the solution of (20)–(22).

EXERCISES

In Exercises 1 through 10, solve the initial-boundary value problem (1)–(4) for the given conditions.

1. $a = 1, l = 1, f(x) = x, F(x, t) = x + t$
2. $a = 1, l = 1, f(x) = x, F(x, t) = xt$
3. $a = 1, l = \pi, f(x) = \sin x, F(x, t) = \cos t$
4. $a = 1, l = \pi, f(x) = x^2, F(x, t) = xe^t$
5. $a = 5, l = 1, f(x) = e^x, F(x, t) = x - 3$
6. $a = 1, l = 1, f(x) = x(1 - x), F(x, t) = \sin t$
7. $a = 3, l = \pi, f(x) = \sin 3x, F(x, t) = \cos t - 3x$
8. $a = 2, l = 1, f(x) = \sin^2 x, F(x, t) = x^2e^{3t}$
9. $a = 2, l = \pi, f(x) = x(\pi - x)^2, F(x, t) = xt^2$
10. $a = 1, l = \pi, f(x) = x + \sin x, F(x, t) = t + \sin x$

In Exercises 11 through 20, solve the initial-boundary value problem (12)–(16) for the given conditions.

11. $c = 1, l = 1, f(x) = x(1 - x), g(x) = 0, F(x, t) = x + t$
12. $c = 1, l = 1, f(x) = x^2(1 - x), g(x) = 0, F(x, t) = xt$
13. $c = 1, l = \pi, f(x) = 0, g(x) = 3, F(x, t) = \cos \pi t$
14. $c = 1, l = \pi, f(x) = x(\pi - x), g(x) = 3, F(x, t) = xe^t$
15. $c = 1, l = 1, f(x) = x(1 - x)^2, g(x) = \pi, F(x, t) = x - 3$
16. $c = 1, l = 1, f(x) = x(1 - x), g(x) = 0, F(x, t) = \sin t$
17. $c = 1, l = \pi, f(x) = 0, g(x) = \pi, F(x, t) = \cos \pi t - 3x$
18. $c = 1, l = 1, f(x) = x^2(1 - x), g(x) = 3, F(x, t) = x^2e^{3t}$
19. $c = 1, l = \pi, f(x) = x(\pi - x)^2, g(x) = 0, F(x, t) = xt^2$
20. $c = 2, l = \pi, f(x) = 0, g(x) = \cos x, F(x, t) = t + \sin x$

21. Apply the method of this section to solve the initial-boundary value problem

$$
\begin{aligned}
u_{tt} - c^2 u_{xx} &= F(x, t), && 0 < x < l, \quad t > 0,\\
u(x, 0) &= f(x), && 0 < x < l,\\
u_t(x, 0) &= g(x), && 0 < x < l,\\
u(0, t) &= 0, && t > 0,\\
u_x(l, t) &= 0, && t > 0.
\end{aligned}
$$

22. Apply the method of this section to solve the initial-boundary value problem

$$
\begin{aligned}
u_{tt} - c^2 u_{xx} &= F(x, t) && 0 < x < l, \quad t > 0,\\
u(x, 0) &= f(x), && 0 < x < l,\\
u_t(x, 0) &= g(x), && 0 < x < l,\\
u_x(0, t) &= 0, && t > 0,\\
u(l, t) &= 0, && t > 0.
\end{aligned}
$$

23. Flexible String: Distributed External Forces If $q(x, t, u, u_t, u_x)$ denotes a distributed external force per unit mass in the positive u direction, then in the development of the equation of motion for the string of Section 11.6 one has the additional force term $\rho(\Delta x)q$. Consequently, the basic partial differential equation is

$$u_{tt} - c^2 u_{xx} = q.$$

The following special cases lend themselves to simple interpretation:

(i) $q = f(x, t)$, an applied force;
(ii) $q = -g$, gravity is acting;
(iii) $q = -ku$, k a constant, a restoring force;

(iv) $q = -ru_t$, $r(> 0)$ a constant, damping due to air resistance;
(v) $q = -ru_t - ku$, a combination of (iii) and (iv) [*Note:* this provides an alternative interpretation of the telegraph equation presented in Exercise 60, Section 11.6];
(vi) $q = -ru_t - ku + f(x, t)$, the telegraph equation with an applied force.

Find the displacement of a string π units long subjected to gravity if $f(x) = 0$, $g(x) = 3$, and $c = 1$.

24. Heat Equation: Source Terms Set up and solve the initial-boundary value problem for the temperature in a rod of length π units that is heated by an electric current so that $F(x, t) = xe^t$ (see Exercise 29, Section 11.7), if the initial temperature in the rod is given by $f(x) = x^2$ and the temperature at the ends is maintained at 0°C. Take $a = 1$.

25. A string is stretched between the points (0, 0) and (1, 0). The string is initially at rest and is subjected to the external force $\pi^2 \sin \pi x$. Find the displacement of the string as a function of the time t and the distance x. Take $c = 1$.

26. A string is stretched between the points (0, 0) and (1, 0). The string is initially at rest and is subjected to the external force $\pi^2 x$. Find the displacement of the string as a function of the time t and the distance x. Take $c = 1$.

11.10 NONHOMOGENEOUS PARTIAL DIFFERENTIAL EQUATIONS: METHOD II

We write the partial differential equation (1) in Problem 1 of Section 11.5 in the form $A[u] = F(x, y)$, where A is a linear operator. Set

$$u(x, y) = v(x, y) + w(x, y), \tag{1}$$

where $v(x, y)$ is to be a new unknown function, and $w(x, y)$ is an unknown function to be determined. Substitution of (1) into Problem 1 yields

PROBLEM 1′

$$\begin{aligned} A[v] &= F(x, t) - A[w], \\ A_1 v(0, y) + A_2 v_x(0, y) &= f_1(y) - A_1 w(0, y) - A_2 w_x(0, y) \\ A_3 v(l, y) + A_4 v_x(l, y) &= f_2(y) - A_3 w(l, y) - A_4 w_x(l, y) \\ A_5 v(x, 0) &= f_3(x) - A_5 w(x, 0) \\ A_6 v_y(x, 0) &= f_4(x) - A_6 w_y(x, 0). \end{aligned}$$

The idea is to choose a specific $w(x, t)$ so that Problem 1′ reduces to a problem that is solvable either by the method of Section 11.9 or by one of the methods in Sections 11.6, 11.7, or 11.8. For a given problem we may have considerable flexibility in the choice of w, since the only criterion is that Problem 1′ be a solvable problem. Once w is chosen and v is calculated, the desired solution u is obtained from Eq. (1); in other words, $u = w + v$.

EXAMPLE 1 Find the solution of the initial-boundary value problem

$$\begin{aligned} u_t - a u_{xx} &= F_1(x, t), \quad 0 < x < l, \quad t > 0, \\ u(0, t) &= f_1(t), \qquad t > 0, \\ u(l, t) &= 0, \qquad t > 0, \\ u(x, 0) &= f_3(x), \qquad 0 < x < l. \end{aligned}$$

Solution Setting $u(x, t) = v(x, t) + w(x, t)$, Problem 1′ becomes

$$\begin{aligned} v_t - av_{xx} &= F_1(x, t) + aw_{xx} - w_t, && 0 < x < l, \quad t > 0, \\ v(0, t) &= f_1(t) - w(0, t), && t > 0, \\ v(l, t) &= -w(l, t), && t > 0, \\ v(x, 0) &= f_3(x) - w(x, 0), && 0 < x < l. \end{aligned}$$

If we can choose $w(x, t)$ so that

$$f_1(t) - w(0, t) = 0 \tag{2}$$

and

$$-w(l, t) = 0, \tag{3}$$

then Problem 1′ will be nothing more than Example 1 of Section 11.9. Condition (3) can be satisfied in many ways, but a simple solution is

$$w(x, t) = (x - l)g(t),$$

where g is to be determined. With this choice, condition (2) becomes

$$f_1(t) + lg(t) = 0,$$

and so

$$g(t) = (-1/l)f_1(t).$$

Hence, a convenient choice for w is

$$w(x, t) = \left(\frac{l - x}{l}\right) f_1(t).$$

Then v is to be a solution of

$$\begin{aligned} v_t - av_{xx} &= F(x, t), && 0 < x < l, \quad t > 0 \\ v(0, t) &= 0, && t > 0, \\ v(l, t) &= 0, && t > 0, \\ v(x, 0) &= f(x), && 0 < x < l, \end{aligned}$$

where

$$F(x, t) = F_1(x, t) - \left(\frac{l - x}{l}\right) f_1'(t) \quad \text{and} \quad f(x) = f_3(x) - \left(\frac{l - x}{l}\right) f_1(0).$$

This latter problem can be solved by the method of Section 11.9, and we can assume that v is now known. Thus, the solution of the original problem is $u = v + w$.

EXAMPLE 2 Solve the boundary value problem

$$\begin{aligned} u_{xx} + u_{yy} &= x^2, & 0<x<l, \quad 0<y<m, \\ u(x, 0) &= f_1(x), & 0<x<l, \\ u(x, m) &= g_1(x), & 0<x<l, \\ u(0, y) &= 0, & 0<y<m, \\ u(l, y) &= 0, & 0<y<m. \end{aligned}$$

Solution Setting $u(x, y) = v(x, y) + w(x, y)$, Problem 1′ becomes

$$\begin{aligned} v_{xx} + v_{yy} &= x^2 - w_{xx} - w_{yy}, & 0<x<l, \quad 0<y<m, \\ v(x, 0) &= f_1(x) - w(x, 0), & 0<x<l, \\ v(x, m) &= g_1(x) - w(x, m), & 0<x<l, \\ v(0, y) &= -w(0, y), & 0<y<m, \\ v(l, y) &= -w(l, y), & 0<y<m. \end{aligned}$$

Choose

$$w(x, y) = \frac{1}{12} x(x^3 - l^3).$$

Then

$$w(0, y) = 0, \; w(l, y) = 0, \; w_{yy} = 0, \; w_{xx} = x^2,$$

and Problem 1′ takes the form

$$\begin{aligned} v_{xx} + v_{yy} &= 0, & 0<x<l, \quad 0<y<m, \\ v(x, 0) &= f(x), & 0<x<l, \\ v(x, m) &= g(x), & 0<x<l, \\ v(0, y) &= 0, & 0<y<m, \\ v(l, y) &= 0, & 0<y<m, \end{aligned}$$

where

$$f(x) = f_1(x) - \frac{1}{12}x(x^3 - l^3) \quad \text{and} \quad g(x) = g_1(x) - \frac{1}{12}x(x^3 - l^3).$$

This latter problem is Example 1 of Section 11.8. Thus we can solve for v by the methods of that section, and

$$u(x, t) = v(x, t) + \tfrac{1}{12} x(x^3 - l^3).$$

REMARK 1 Note that any choice of w that has the property that $x^2 - w_{xx} - w_{yy} = 0$ will reduce Problem 1′ to a solvable problem. For example, the choice

$$w(x, y) = \tfrac{1}{12}x^4 + \alpha x + \beta y + \gamma,$$

where α, β, and γ are any constants, reduces Problem 1′ to

$$\begin{aligned} v_{xx} + v_{yy} &= 0, \quad & 0<x<l, \quad 0<y<m, \\ v(x, 0) &= f(x), & 0<x<l, \\ v(x, m) &= g(x), & 0<x<l, \\ v(0, y) &= h(y), & 0<y<m, \\ v(l, y) &= k(y), & 0<y<m, \end{aligned}$$

with

$$\begin{aligned} f(x) &= f_1(x) - \tfrac{1}{12}x^4 - \alpha x - \gamma, \\ g(x) &= g_1(x) - \tfrac{1}{12}x^4 - \alpha x - \beta m - \gamma, \\ h(y) &= -\beta y - \gamma \end{aligned}$$

and

$$k(y) = -\tfrac{1}{12}l^4 - \alpha l - \beta y - \gamma.$$

This latter problem is solvable. (See Example 3, Section 11.8.)

REMARK 2 There is only one solution, u, of the original boundary value problem of Example 2. Therefore, different choices for w (and correspondingly different solutions v) do not give rise to different solutions, but rather to alternative versions of the same function.

REMARK 3 The utility of the method of this section hinges upon our ability to spot appropriate forms for w. Very often the question of whether this task is easy or not is dictated by the form of F, f_1, f_2, f_3, or f_4.

EXERCISES

In Exercises 1 through 5, solve the initial-boundary value problem of Example 1 for each of the given cases. Take $a = l = 1$.

1. $F_1(x, t) = 2t(1 - x), f_1(t) = t^2, f_3(x) = x$
2. $F_1(x, t) = (x - 1)\sin t, f_1(t) = \cos t, f_3(x) = x(1 - x)$
3. $F_1(x, t) = t, f_1(t) = \frac{1}{2}t^2, f_3(x) = x$
4. $F_1(x, t) = xe^t, f_1(t) = -e^t, f_3(x) = 3x$
5. $F_1(x, t) = 5xe^{3t}, f_1(t) = 2t, f_3(x) = x$

In Exercises 6 through 10, solve the boundary value problem of Example 2 for each of the given cases.

6. $f_1(x) = \frac{1}{12}x^4,\ g_1(x) = 0$

7. $f_1(x) = \frac{1}{12}x^4,\ g_1(x) = \frac{1}{12}x(x^3 - 1)$

8. $f_1(x) = 0,\ g_1(x) = x$

9. $f_1(x) = x,\ g_1(x) = 0$

10. $f_1(x) = 1 - x,\ g_1(x) = x$

11. Solve the boundary value problem

$$\begin{aligned}
u_{xx} + u_{yy} &= (2 - x^2)\sin y, && 0<x<1, \quad 0<y<\pi \\
u(x, 0) &= 0, && 0<x<1, \\
u(x, \pi) &= 0, && 0<x<1, \\
u(0, y) &= h(y), && 0<y<\pi, \\
u(1, y) &= k(y), && 0<y<\pi
\end{aligned}$$

by finding an appropriate function $w(x, y)$ so that the function $v(x, y) = u(x, y) - w(x, y)$ solves a boundary value problem for which the partial differential equation is homogeneous. [*Hint:* try $w(x, y) = Ax^2 \sin y$, where A is to be determined.]

12. Solve the initial-boundary value problem

$$\begin{aligned}
u_t - 2u_{xx} &= \left(\frac{\pi - x}{\pi}\right)\cos t + \frac{xt}{\pi}(2 + \pi t), && 0<x<\pi, \quad t>0, \\
u(0, t) &= \sin t, && t>0, \\
u(\pi, t) &= t^2, && t>0, \\
u(x, 0) &= x(\pi - x)^2, && 0<x<\pi
\end{aligned}$$

by finding an appropriate function $w(x, t)$ so that $v(x, t) = u(x, t) - w(x, t)$ solves an initial-boundary value problem such that $v(0, t) = v(\pi, t) = 0$.

13. Solve the initial-boundary value problem

$$\begin{aligned}
u_{tt} - u_{xx} &= \cos \pi t + \frac{10}{\pi}(\pi - x) + \frac{xe^t}{\pi}, && 0<x<\pi, \quad t>0, \\
u(x, 0) &= \frac{x}{\pi}, && 0<x<\pi, \\
u_t(x, 0) &= \frac{x}{\pi} + 3, && 0<x<\pi, \\
u(0, t) &= 5t^2, && t>0, \\
u(\pi, t) &= e^t, && t>0
\end{aligned}$$

by finding an appropriate function $w(x, t)$ so that $v(x, t) = u(x, t) - w(x, t)$ solves an initial-boundary value problem such that $v(0, t) = v(\pi, t) = 0$.

14. Torsion of a Beam The two-dimensional nonhomogeneous Laplace equation

$$u_{xx} + u_{yy} = f(x, y) \tag{4}$$

is known as the (two-dimensional) *Poisson equation.* If a beam of uniform cross section has its axis coincident with the z-axis, then linear elasticity theory shows that the stress function, $u(x, y)$, for the beam satisfies Eq. (4) with $f(x, y) = -2$ and $u = 0$ on the boundary of the area of intersection of the beam with the xy-plane. Find the stress function of a square beam of side π units. [*Hint:* The original rectangle is $-\pi/2 \le x \le \pi/2$, $-\pi/2 \le y \le \pi/2$. For convenience of the eigenvalue problem, set $s = x + \pi/2$, $t = y + \pi/2$, and note that Poisson's equation becomes $u_{ss} + u_{tt} = -2$. Solve the new boundary value problem as a first step.]

15. Assume a section of the earth's crust to be a rod with one end at the surface of the earth (considered to be at $x = 0$) and the other end (considered to be at $x = l$) inside the earth at such a depth that the temperature at that end is fixed (taken to be 0°C). Consider that the surface of the earth is warmed by the sun so that the temperature, u, at the surface is given by $28 \cos \dfrac{\pi}{12}(t - 12)$ (the time t is measured in hours; 1 AM corresponds to $t = 1$, and midnight corresponds to $t = 24$). If the initial temperature distribution in the earth's crust is $f(x) = 28\left(\dfrac{x - l}{l}\right)$, set up and solve the initial-boundary value problem for the temperature in the earth's crust when $a = l = 1$. (Assume that the effect of time units in hours rather than seconds has already been accounted for and that no further modifications are necessary.)

16. Suppose that a circular shaft (refer to Exercise 51, Section 11.6) has the end $x = 0$ fixed and is initially at rest in its equilibrium position. Assume that the shaft undergoes torsional vibrations due to a periodic rotation at the end $x = l$ (take $l = 1$) of the form $f(t) = \cos t$. Set up and solve the initial-boundary value problem for the angular displacement. Take $c = 1$.

17. The following initial-boundary value problem corresponds to a special case of the *telegraph equation* (Exercise 60, Section 11.6).

$$\begin{aligned}
u_{tt} + a^2 u &= u_{xx}, && 0 < x < l, \quad t > 0\\
u(x, 0) &= 0, && 0 < x < l,\\
u_t(x, 0) &= 0, && 0 < x < l,\\
u(0, t) &= t^3/3!, && t > 0,\\
\rho u_x(l, t) &= u(l, t), && t > 0,
\end{aligned}$$

where a, ρ are constants. Set

$$u(x, t) = v(x, t) + \left(\frac{l - x}{l}\right)^2 \frac{t^3}{3!}$$

and determine an equation for the eigenvalues of the initial-boundary value problem for $v(x, t)$.

18. **Loss of Heat through the Sides of a Pipe** Suppose we have a cylindrical pipe filled with a hot fluid (of constant temperature), and we wish to investigate the loss of heat through the sides of the pipe. The pipe can be considered a hollow cylinder of inner radius r_1, outer radius r_2; therefore, the temperature *in the pipe* satisfies the initial value problem

$$u_t = \frac{a}{r}(ru_r)_r, \quad r_1 < r < r_2, \quad t > 0$$

$$\begin{aligned} u(r_1, t) &= A, && t > 0, \\ u(r_2, t) &= B, && t > 0, \\ u(r, 0) &= f(r), && r_1 < r < r_2, \end{aligned}$$

where A is the temperature of the fluid, B is the temperature of the air (or medium) surrounding the pipe, and $f(r)$ is the initial distribution of temperature in the pipe. For the case $A = 100°\text{C}$, $B = 0°\text{C}$, $r_1 = 100$ cm, $r_2 = 101$ cm, and $f(r) = 100(101 - r)$, take $a = 1$ and show that the substitution

$$u(r, t) = v(r, t) + 100(101 - r)$$

produces a nonhomogeneous equation (for v) with homogeneous initial-boundary conditions (the solution of which involves Bessel functions).

REVIEW EXERCISES

In each of Exercises 1 through 4, state the order of the partial differential equation, state whether it is linear or quasilinear, and, if it is linear, indicate whether it is homogeneous or nonhomogeneous and whether it has constant coefficients or variable coefficients.

1. $u_{xxyz} - u_{xxy} + u_{zy} + 3u = 0$
2. $u_{xx} - 5y^2u_{yy} = 3xy$
3. $uu_{xx} + 5u_{yy} - 2(u_x)^2 + 2u^2u_y = 0$
4. $xu_t - 3u_{xx} + 5u_x = 0$

In Exercises 5 and 6, assume that u is a function of four variables x, y, z and t. Integrate the equation to obtain the general solution.

5. $u_{xy} = z^2 + 2xy - t$
6. $u_{xyt} + \dfrac{1}{t}u_{xy} = 2xy$ [*Hint:* set $v = u_{xy}$.]
7. Show that $u = 3xy + x^2$ is a particular solution of $u_x + u_y = 5x + 3y$.
8. Show that $u = x^3 + \cos(x + 3y)$ is a particular solution of $9u_{xx} - u_{yy} = 54x$.

9. Suppose that $A[u] = u_{xxy} - 5u_{xy} + 2u_y$ and that u_1 is a particular solution of the equation $A[u] = 8x - 9y$ and that u_2 is a particular solution of the equation $A[u] = 4y - 3x + \cos(xy)$. Find a particular solution to the equation $A[u] = 5(x - y) + \cos xy$.

10. Suppose that $A[u] = u_{xx} + 3u_{xy} - 7u_y + 8u$ and that u_1 is a particular solution of the equation $A[u] = e^x \cos y$ and that u_2 is a particular solution of the equation $A[u] = e^{-x} \cos y$. Find a particular solution to the equation $A[u] = \cosh x \cos y$.

In each of Exercises 11 through 14, find a particular solution in the form $e^{\lambda x + \mu y}$.

11. $u_{xx} - u_{yy} = 0$

12. $u_{xx} - u_{xy} + u_x - u_y = 0$

13. $u_{xy} - u_x + u_y - 3u = 0$

14. $u_y - u_x = 0$

In Exercises 15 and 16, assume a solution in the form $u = X(x)Y(y)$ and determine the ordinary differential equations that X and Y satisfy.

15. $3u_x + 8u_y = 2(x + y)u$

16. $u_{xy} - 3u = 0$

In Exercises 17 through 20, classify the equation as hyperbolic, parabolic, or elliptic. Find the general solution.

17. $u_{xx} + u_{yy} = 0$

18. $u_{xx} - 12u_{xy} + 36u_{yy} = 0$

19. $u_{xx} + 4u_{xy} - 5u_{yy} = 0$

20. $u_{xy} + u_{yy} = 0$

21. Solve the following initial-boundary value problem.

$$\begin{aligned}
u_t - u_{xx} &= 0, && 0 < x < 1, \quad t > 0\\
u(x, 0) &= 3 - 3x, && 0 < x < 1,\\
u(0, t) &= 3, && t > 0,\\
u(1, t) &= 1, && t > 0.
\end{aligned}$$

22. Solve the following initial-boundary value problem

$$\begin{aligned}
u_{tt} - 4u_{xx} &= 0, && 0 < x < 1, \quad t > 0,\\
u(x, 0) &= 0, && 0 < x < 1,\\
u_t(x, 0) &= x, && 0 < x < 1,\\
u(0, t) &= 1, && t > 0,\\
u(1, t) &= 0, && t > 0.
\end{aligned}$$

23. Solve the following boundary value problem.

$$\begin{aligned}
u_{xx} + u_{yy} &= 0, && 0 < x < 1, \quad 0 < y < 1,\\
u(x, 0) &= x^2, && 0 < x < 1,\\
u(x, 1) &= 0, && 0 < x < 1,
\end{aligned}$$

$$u(0, y) = \cos y, \quad 0 < y < 1,$$
$$u(1, y) = 0, \quad 0 < y < 1.$$

24. Using the method discussed in Section 11.9, solve the following initial-boundary value problem.

$$u_{tt} - u_{xx} = \frac{1}{\sqrt{2}} xt, \qquad 0 < x < 1, \quad t > 0,$$
$$u(x, 0) = \sqrt{2} \sin \pi x, \quad 0 < x < 1,$$
$$u_t(x, 0) = \sqrt{2} \sin 5\pi x, \quad 0 < x < 1,$$
$$u(0, t) = 0, \qquad t > 0,$$
$$u(1, t) = 0, \qquad t > 0.$$

25. Using the method discussed in Section 11.9, solve the following initial-boundary value problem.

$$u_t - u_{xx} = \frac{1}{\sqrt{2}} x e^{\pi^2 t}, \qquad 0 < x < 1, \quad t > 0,$$
$$u(x, 0) = \sqrt{2} \sin \pi x, \quad 0 < x < 1,$$
$$u(0, t) = 0, \qquad t > 0,$$
$$u(1, t) = 0, \qquad t > 0.$$

26. Solve the nonhomogeneous boundary value problem.

$$u_{xx} + u_{yy} = (2 - \pi^2 x^2) \cos \pi y + \left(2 - \frac{\pi^2 x^2}{4}\right) \sin \frac{\pi}{2} y, \quad 0 < x < 1, \quad 0 < y < 1,$$
$$u(x, 0) = x^2, \qquad 0 < x < 1,$$
$$u(x, 1) = 0, \qquad 0 < x < 1,$$
$$u(0, y) = y, \qquad 0 < y < 1,$$
$$u(1, y) = \sin \frac{\pi}{2} y + \cos \pi y, \qquad 0 < y < 1$$

by finding an appropriate function $w(x, y)$ so that the substitution $v(x, y) = u(x, y) - w(x, y)$ leads to a boundary value problem for v that contains a homogeneous partial differential equation. [*Hint:* try w in the form $f(x)[\cos \pi y + \sin \frac{\pi}{2} y]$.]

27. Longitudinal Vibrations in a Bar *Longitudinal* vibrations, $u(x, t)$, of a bar with one end free (see Exercise 51, Section 11.6) and the other end (taken to be $x = l$) subjected to a constant force, p, leads to the initial-boundary value problem.[13]

[13] I. N. Bronshtein and K. A. Semendyayev, *A Guidebook to Mathematics* (New York: Springer-Verlag, 1973), p. 572.

$$
\begin{aligned}
u_{tt} - a^2 u_{xx} &= 0, && 0 < x < l, \quad t > 0,\\
u(x, 0) &= f(x), && 0 < x < l,\\
u_t(x, 0) &= g(x), && 0 < x < l,\\
u_x(0, t) &= 0, && t > 0,\\
u_x(l, t) &= kp, && t > 0,
\end{aligned}
$$

where k is a constant. Describe how to find u as a function of x and t.

28. A harp string[14] is plucked so that its initial velocity is zero and its initial shape is

$$
u(x, 0) = \begin{cases} \dfrac{20h}{9l}x, & 0 < x < \dfrac{9l}{20},\\[2mm] \dfrac{20h}{l}\left(\dfrac{l}{2} - x\right), & \dfrac{9l}{20} < x < \dfrac{11l}{20},\\[2mm] \dfrac{20h}{9l}(x - l), & \dfrac{11l}{20} < x < l. \end{cases}
$$

Find u as a function of x and t.

[14]This is Exercise 2 in Philip M. Morse and K. Uno Ingard, *Theoretical Acoustics* (New York: McGraw-Hill Book Co., 1968), p. 169. Used with the permission of McGraw-Hill Book Company.

1. Adding y_{2k+1} to both sides, we also find

$$\begin{aligned} y_1 - y_2 + \cdots + y_{2k-1} - y_{2k} + y_{2k+1} &= 1 - y_{2k-1} + y_{2k+1} \\ &= 1 - y_{2k-1} + y_{2k} + y_{2k-1} \\ &= 1 + y_{2k}. \end{aligned}$$

From these two results the desired identity follows for all k (even or odd).

Section 10.2

1. True. **3.** False. **5.** True. **7.** False. **9.** True. **11.** True.

13. See Figure 10.1(a). **15.** See Figure 10.1(b).

17. Compute $(af + bg)(x + T)$ and $(fg)(x + T)$.

19. $p > 0$; the period is $2\pi/\sqrt{p}$.

Section 10.3

1. $f(x) \sim -\dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin n\pi x.$

3. $f(x) \sim \dfrac{1}{2} + \dfrac{1}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx.$

5. $f(x) \sim \pi - 2\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}\sin nx.$

7. $f(x) \sim \dfrac{4\pi^2}{3} + \displaystyle\sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right).$

9. $f(x) \sim 1 + \cos 2x.$

11. $f(x) \sim \sin 2x.$

13. $f(x) \sim 2\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{6-n^2\pi^2}{n^3}\sin nx.$

15. $f(x) \sim \dfrac{2}{\pi} - \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}\cos nx.$

17. $f(x) \sim \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} \sin\frac{n\pi}{2} \cos\frac{n\pi x}{2}$.

19. $f(x) \sim \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin\frac{n\pi x}{l}$.

Section 10.4
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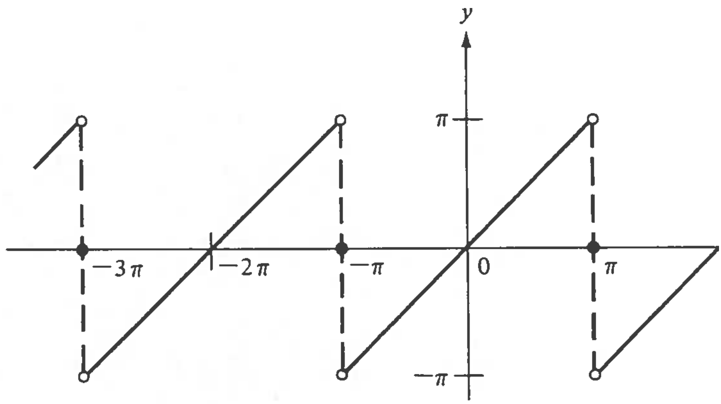
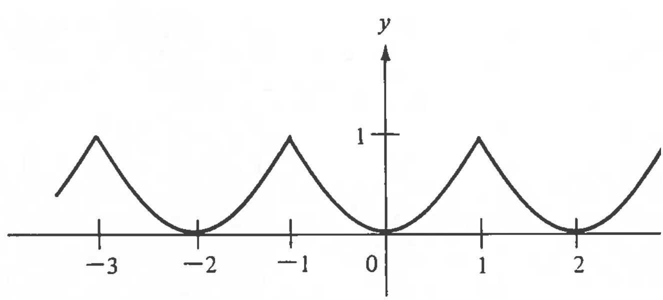
1. $f(x) \sim \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx.$

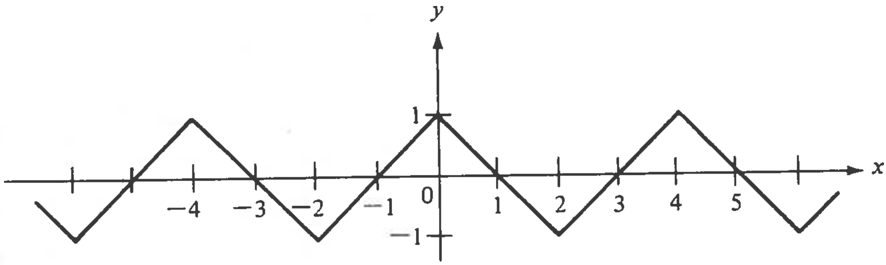


3. $f(x) \sim \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi x.$

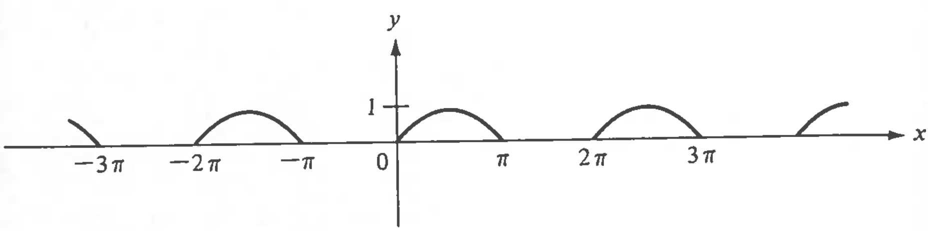


Section 10.4

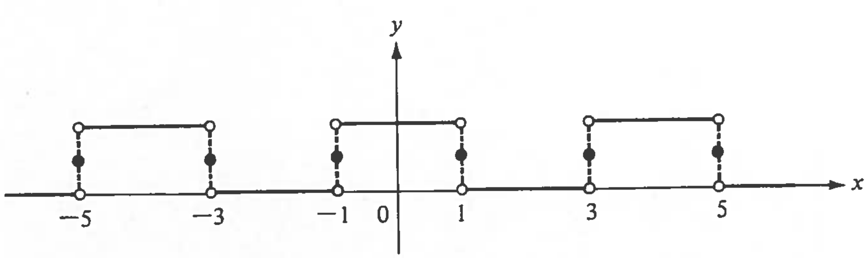
5. $f(x) \sim \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} \cos \frac{n\pi x}{2}$.



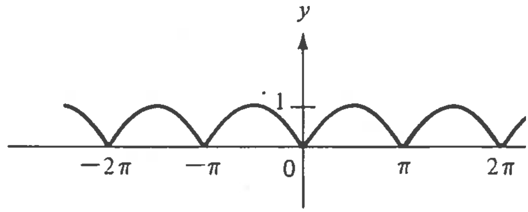
7. $f(x) \sim \frac{1}{\pi} + \frac{1}{2}\sin x + \frac{1}{\pi} \sum_{n=2}^{\infty} \frac{1+(-1)^n}{1-n^2} \cos nx.$



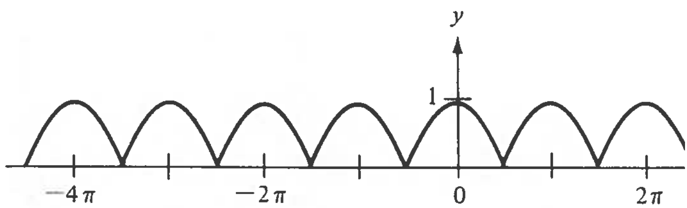
9. $f(x) \sim \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2} \cos \frac{n\pi x}{2}$.



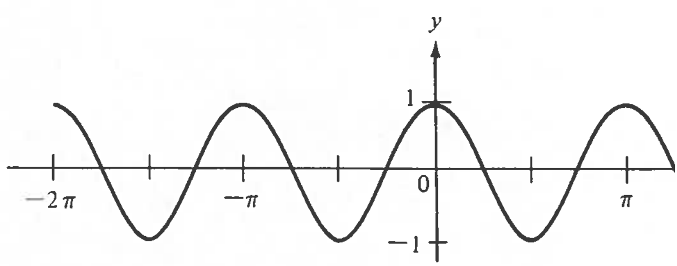
11. $f(x) = \frac{1}{2} - \frac{1}{2}\cos 2x.$



13. $f(x) = \frac{1}{2} + \frac{1}{2}\cos 2x.$



15. $f(x) = \cos 2x.$



17. Use Exercise 1.

19. In the identity of Exercise 18, set $x = 1$ and $x = 0$.

21. False. 23. True.

25. False. 27. False.

29. True.

Section 10.5

1. True. 3. False. 5. False. 7. True. 9. False.

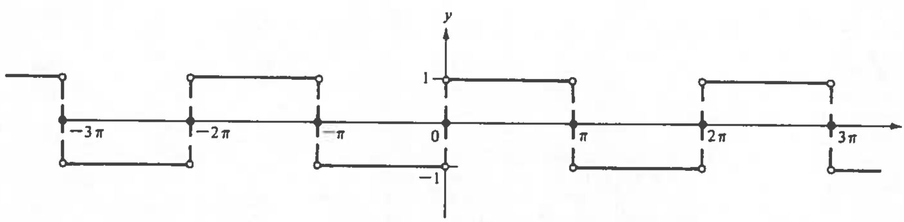
11. Even; $f(x) \sim \frac{1}{2} + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos n\pi x.$

13. Odd; $f(x) \sim \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \sin nx.$

15. Odd; $f(x) \sim \frac{2}{\pi^3} \sum_{n=1}^{\infty} (-1)^n \frac{6 - n^2\pi^2}{n^3} \sin n\pi x.$

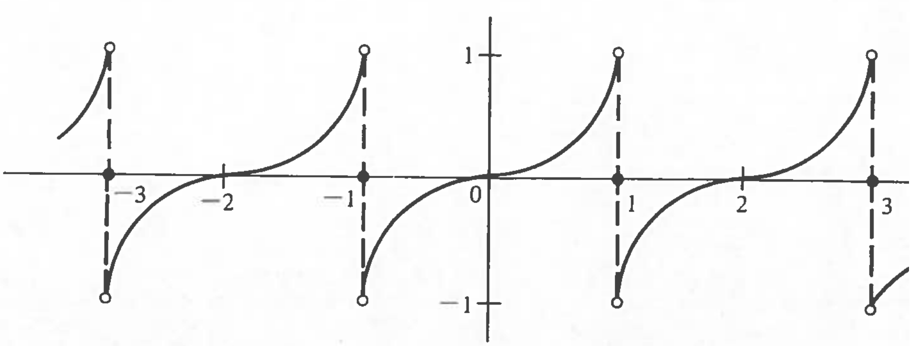
17. Odd; $f(x) \sim \sin 2x.$

19. $f(x) \sim \frac{4}{\pi} (\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots).$

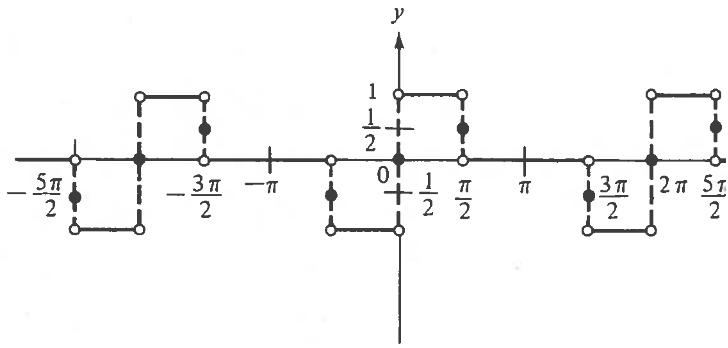


21. $f(x) \sim 2(\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \cdots)$. For the graph of the function to whi the series converges, see Exercise 1, Section 10.4.

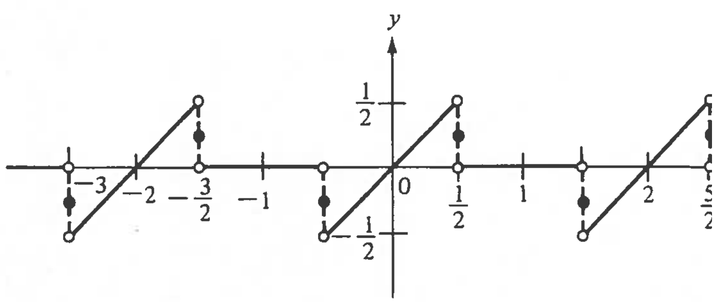
23. $f(x) \sim -2 \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{n\pi} + \frac{2[1 - (-1)^n]}{n^3\pi^3} \right] \sin n\pi x.$



25. $f(x) \sim \dfrac{2}{\pi} \sum_{n=1}^{\infty} \dfrac{1}{n}\left(1 - \cos\dfrac{n\pi}{2}\right) \sin nx.$



27. $f(x) \sim \sum_{n=1}^{\infty} \left(\dfrac{2}{n^2\pi^2} \sin\dfrac{n\pi}{2} - \dfrac{1}{n\pi}\cos\dfrac{n\pi}{2}\right) \sin n\pi x.$



REVIEW EXERCISES CHAPTER 10
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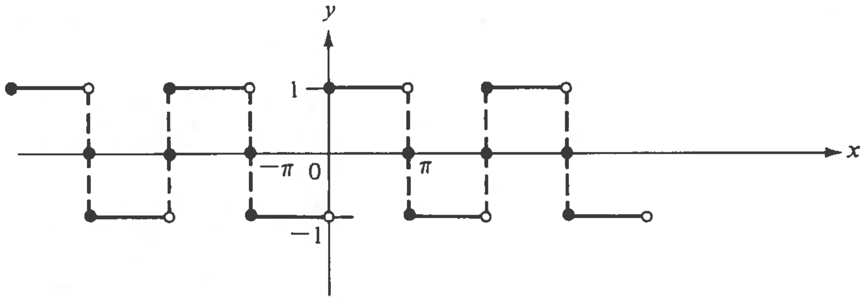
1. $f(x) \sim -3 - \dfrac{36}{\pi^2} \sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^2} \cos\dfrac{n\pi x}{3}$.

3. $f(x) \sim \dfrac{3\sqrt{3}}{2\pi} + \dfrac{3\sqrt{3}}{\pi} \sum_{n=1}^{\infty} \dfrac{(-1)^n}{1 - 9n^2} \cos nx.$

5. $f(x) \sim 1 + \dfrac{e - e^{-1}}{2} + (e - e^{-1}) \sum_{n=1}^{\infty} \dfrac{(-1)^n}{1 + n^2\pi^2} (\cos n\pi x - n\pi \sin n\pi x)$

7. $P(t) \sim P_0(1 - e^{-1}) + 2P_0(1 - e^{-1}) \sum_{n=1}^{\infty} \dfrac{1}{1 + 4n^2\pi^2} (\cos 200n\pi t + 2n\pi$

9. $f(x) \sim \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right).$



11. $f(x) \sim 1 + \cos x.$

13. Find the Fourier series of the function $f(x) = |x|, -1 \le x < 1.$

15. $f(x) \sim \left(1 + \frac{\pi}{2}\right) + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos nx.$

17. $f(x) \sim \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2 - 1}\cos 2nt.$

Section 11.2

1. third order, homogeneous, variable coefficients.

2. fourth order, homogeneous, constant coefficients.

5. second order, homogeneous, constant coefficients.

7. first order, nonhomogeneous, variable coefficients.

9. first order, nonhomogeneous, constant coefficients.

11. $u(x, y) = f(x).$ 13. $u(x, y) = x^3 + 4xy + f(y).$

15. $u(x, y) = f(x) + g(y).$ 17. $u(x, y) = yf(x) + g(x).$

19. $u(x, y) = x \sin y + ye^x + f(x) + g(y).$

21. $u(x, y, z) = xyz + f(y, z) + g(x, y).$

23. $u(x, y, z) = f(x, z) + xg(y, z) + h(y, z)$.

25. $u(x, y, z) = f(x, y) + g(x, z) + xh(y, z) + k(y, z)$.

27. $u(x, y, z) = yz + f(x, y)$.

29. $u(x, y, z) = \frac{1}{2}yz^2 + \frac{3}{2}xz^2 + zf_1(x, y) + f_2(x, y)$.

Section 11.3
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25. $u(x, y) = \frac{1}{2}xy^2 + f(x)$.

27. $u(x, y) = xf(y) + g(y)$.

29. $u(x, y) = \frac{1}{12}x^2y^3 + yf(x) + g(x) + h(y)$.

31. $u(x, y) = y \sin x - e^y + f(x)$.

33. $u_p = e^{\lambda x + (1-3\lambda)y}$, λ arbitrary.

35. $u_p = e^{[4-(3/2)\mu]x + \mu y}$, μ arbitrary.

37. $u_p = e^{[(3/5)\mu + 2/5]x + \mu y}$, μ arbitrary.

39. $u_p = e^{(-\mu-2)x + \mu y}$ or $u_p = e^{(-\mu-1)x + \mu y}$, μ arbitrary.

41. $u_p = e^{(\mu-4)x + \mu y}$ or $u_p = e^{(\mu-1)x + \mu y}$, μ arbitrary.

43. $u_p = e^{\lambda x + (2\lambda+3)y}$ or $u_p = e^{\lambda x + (2\lambda-1)y}$, λ arbitrary.

45. $u_p = e^{\lambda x + [1-(2/3)\lambda]y}$ or $u_p = e^{\lambda x + [-2/3-(2/3)\lambda]y}$, λ arbitrary.

47. $u_p = e^{x+y}$, corresponding to $\lambda = \mu = 1$.

49. (a) $u_p = -3x^2 - 6x - 6$. (b) $u_p = \cos y +$ s
(c) $u_p = 2x^2 - 4x - 3y^2 - 6y$. (d) $u_p = \frac{10}{53}e^{3x+4y}$

Section 11.4
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1. $X' - \lambda X = 0$, $3Y' - \lambda Y = 0$.

3. $2X' - \lambda X = 0$, $5Y' + \lambda Y = 0$.

5. $X' + (1 - \lambda)X = 0$, $Y' + \lambda Y = 0$.

7. $X'' - \lambda X = 0,\ Y' - \lambda Y = 0.$

9. $X'' - \lambda X = 0,\ Y'' - \lambda Y = 0.$

11. $X'' + X' - \lambda X = 0,\ Y'' + Y' + \lambda Y = 0.$

13. $X'' - X' - \lambda X = 0,\ Y'' - Y' - \lambda Y = 0.$

15. $X'' - X' - \lambda X = 0,\ Y' - \lambda Y = 0.$

17. $5X'' + 6X' - \lambda X = 0,\ 3Y' - \lambda Y = 0.$

19. $2X'' + (1 - \lambda)X = 0,\ 3Y'' + \lambda Y = 0.$

21. $aX'' + dX' + (f - \lambda)X = 0,\ cY'' + eY' + \lambda Y = 0.$

23. $X' - \lambda X = 0,\ Y' - \mu Y = 0,\ Z' + (\lambda - \mu)Z = 0.$

25. $X' - \lambda X = 0,\ 2Y' - \mu Y = 0,\ 2Z' - (\lambda + \mu)Z = 0.$

27. $X'' - \lambda X = 0,\ Y'' + \mu Y = 0,\ Z'' + (\lambda - \mu)Z = 0.$

29. $X'' - \lambda X = 0,\ Y'' - \mu Y = 0,\ Z'' + (\lambda - \mu)Z = 0.$

31. $2X'' + (1 - \lambda)X = 0,\ Y'' - \mu Y = 0,\ Z'' + (\lambda - \mu)Z = 0.$

41. Elliptic. $u(x, y) = f\left(x + \left[-\frac{1}{4} + i\frac{\sqrt{3}}{12}\right]y\right) + g\left(x + \left[-\frac{1}{4} - i\frac{\sqrt{3}}{12}\right]y\right).$

43. Elliptic. $u(x, y) = f\left(x + \left[-\frac{1}{4} + i\frac{\sqrt{3}}{4}\right]y\right) + g\left(x + \left[-\frac{1}{4} - i\frac{\sqrt{3}}{4}\right]y\right).$

45. Hyperbolic. $u(x, y) = f\left(x + \frac{-9 + \sqrt{65}}{8}y\right) + g\left(x + \frac{-9 - \sqrt{65}}{8}y\right).$

47. Hyperbolic. $u(x, y) = f\left(x + \frac{-5 + \sqrt{21}}{2}y\right) + g\left(x + \frac{-5 - \sqrt{21}}{2}y\right).$

49. Parabolic. $u(x, y) = f(x - \frac{1}{4}y) + (\alpha x + \beta y)g(x - \frac{1}{4}y)$, α, β arbitrary.

51. Hyperbolic. $u(x, y) = f(x + 2y) + g(x - 3y).$

53. Hyperbolic. $u(x, y) = f(x + 3y) + g(x + 7y).$

55. Elliptic. $u(x, y) = f\left(x + \frac{-2 + i}{5}y\right) + g\left(x + \frac{-2 - i}{5}y\right).$

57. Hyperbolic. $u(x, y) = f(x + \frac{1}{4}y) + g(x - y)$.

59. Parabolic. $u(x, y) = f(x - \frac{1}{3}y) + (\alpha x + \beta y)g(x - \frac{1}{3}y)$, α, β arbitrar

61. (b) $X'' + \left[\dfrac{8m\pi^2E}{h^2} - \lambda\right]X = 0,\ Y'' + (\lambda - \mu)Y = 0,\ Z'' + \mu Z =$

63. $(M^2 - 1)X'' - \lambda X = 0,\ Y'' - \lambda Y = 0.$

65. $T'' + (a^2 - \lambda)T = 0,\ X'' - \lambda X = 0.$

$\lambda < 0$: $T(t) = c_1 \cos \sqrt{a^2 - \lambda}\, t + c_2 \sin \sqrt{a^2 - \lambda}\, t$
$X(x) = d_1 \cos \sqrt{-\lambda}\, x + d_2 \sin \sqrt{-\lambda}\, x.$

$\lambda = 0$: $T(t) = c_1 \cos at + c_2 \sin at$
$X(x) = d_1 + d_2 x.$

$0 < \lambda < a^2$: $T(t) = c_1 \cos \sqrt{a^2 - \lambda}\, t + c_2 \sin \sqrt{a^2 - \lambda}\, t$
$X(x) = d_1 e^{\sqrt{\lambda} x} + d_2 e^{-\sqrt{\lambda} x}$

$\lambda = a^2$: $T(t) = c_1 + c_2 t$
$X(x) = d_1 e^{ax} + d_2 e^{-ax}$

$\lambda > a^2$: $T(t) = c_1 e^{\sqrt{\lambda - a^2}\, t} + c_2 e^{-\sqrt{\lambda - a^2}\, t}$
$X(x) = d_1 e^{\sqrt{\lambda} x} + d_2 e^{-\sqrt{\lambda} x}$

67. $u(x, y) = \sqrt{\lambda}\, x + \sqrt{1 - \lambda}\, y + c, \quad (0 < \lambda < 1).$

Section 11.6
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1. $$u(x, t) = \sum_{n=1}^{\infty} \frac{8}{\pi(2n - 1)^3} \cos (2n - 1)t \sin (2n - 1)x.$$

3. $$u(x, t) = \sum_{n=1}^{\infty} \left(\frac{-4}{n^3}\right) [1 + 2(-1)^n] \cos nt \sin nx.$$

5. $$u(x, t) = \sum_{n=1}^{\infty} \frac{12}{(2n - 1)^2\pi} \sin (2n - 1)t \sin (2n - 1)x.$$

7. $$u(x, t) = \sum_{n=1}^{\infty} \frac{4A}{(2n - 1)^2\pi^2} \sin (2n - 1)\pi t \sin (2n - 1)\pi x.$$

9. $$u(x, t) = \sum_{n=1}^{\infty} \left[\frac{8}{(2n - 1)^3\pi} \cos (2n - 1)t + \frac{12}{(2n - 1)^2\pi} \sin (2n - 1)t\right] \text{s}$$

11. $$u(x,t)=\sum_{n=1}^{\infty}\frac{3}{2n^3}\cos 2nt\sin 2nx + \sum_{n=1}^{\infty}\left[\frac{4}{(2n-1)^3}\cos(2n-1)t+\frac{4}{(2n-1)^2}\sin(2n-1)t\right]\sin(2n-1)x.$$

13. $$u(x,t)=\cos 20t\sin 5t+\sum_{n=1}^{\infty}\frac{1}{\pi[(2n-1)^2-4]}\sin 4(2n-1)t\sin(2n-1)x.$$

15. $$u(x,t)=\sum_{n=1}^{\infty}\frac{-1}{n^2}\cos 2nt\cos 2nx.$$

17. $$u(x,t)=\sum_{n=1}^{\infty}\frac{-\pi}{2n^2}\cos 2nt\cos 2nx + \sum_{n=1}^{\infty}\left[\frac{24}{(2n-1)^4\pi}+\frac{2\pi}{(2n-1)^2}\right]\cos(2n-1)t\cos(2n-1)x.$$

19. $u(x,t)=0.$ 21. $u(x,t)=0.$

23. $$u(x,t)=\sum_{n=1}^{\infty}\frac{-1}{n^2}\cos 2nt\cos 2nx.$$

25. $$u(x,t)=\sum_{n=1}^{\infty}\frac{-\pi}{2n^2}\cos 2nt\cos 2nx + \sum_{n=1}^{\infty}\left[\frac{24}{(2n-1)^4\pi}-\frac{2\pi}{(2n-1)^2}\right]\cos(2n-1)t\cos(2n-1)x.$$

27. $$u(x,t)=\frac{1}{8}\sin 8t\cos 2t+\sum_{n=1}^{\infty}\frac{-20}{[4n^2-25]\pi}\cos 4nt\cos 2nx.$$

29. $$u(x,t)=\sum_{n=1}^{\infty}\left[\alpha_n\cos\frac{(2n-1)\pi ct}{2l}+\beta_n\sin\frac{(2n-1)ct}{2l}\right]\sin\frac{(2n-1)\pi x}{2l},$$

where

$$\alpha_n=\frac{2}{l}\int_0^l f(x)\sin\frac{(2n-1)\pi x}{2l}\,dx$$

and

$$\beta_n=\frac{4}{(2n-1)\pi c}\int_0^l g(x)\sin\frac{(2n-1)\pi x}{2l}\,dx.$$

33. $$u(x,t)=\frac{3x}{\pi}+\sum_{n=1}^{\infty}\frac{3}{n\pi}\cos 2nt\sin 2nx + \sum_{n=1}^{\infty}\left[\frac{8}{(2n-1)^3\pi}-\frac{6}{(2n-1)\pi}\right]\cos(2n-1)t\sin(2n-1)x.$$

35. $$u(x,t) = 2 + \sum_{n=1}^{\infty} \frac{-3}{2n^3}\cos 2nt \sin 2nx + \sum_{n=1}^{\infty}\left[\frac{8}{(2n-1)^3} - \frac{8}{(2n-1)\pi}\right]\cos(2n-1)t \sin$$

37. $$u(x,t) = x + \frac{10(\pi - x)}{\pi} + \sum_{n=1}^{\infty}\frac{\pi - 10}{n\pi}\cos 2nt \sin 2nx + \sum_{n=1}^{\infty}\left\{\left[\frac{2(10-\pi)}{(2n-1)\pi} - \frac{40}{\pi}\right]\cos(2n-1)t + \frac{12}{(2n-1)^2\pi}\sin\right.$$

39. $$u(x,t) = 7 - 5x + \sum_{n=1}^{\infty}\frac{-5}{n\pi}\cos 2n\pi t \sin 2n\pi x + \sum_{n=1}^{\infty}\left[\frac{-18}{(2n-1)\pi}\cos(2n-1)t + \frac{4A}{(2n-1)^2\pi^2}\sin\right.$$

41. $$u(x,t) = \frac{5x}{\pi} + \sum_{n=1}^{\infty}\frac{5}{n\pi}\cos 2nt \sin 2nx + \sum_{n=1}^{\infty}\left\{\left[\frac{8}{(2n-1)^3\pi} - \frac{10}{(2n-1)\pi}\right]\cos(2n -\right.$$
$$+ \frac{12}{(2n-1)^2\pi}\sin$$

43. $$u(x,t) = \frac{8x}{\pi} + \sum_{n=1}^{\infty}\left[\frac{3}{2n^3} + \frac{8}{n\pi}\right]\cos 2nt \sin 2nx + \sum_{n=1}^{\infty}\left\{\left[\frac{4}{(2n-1)^3} - \frac{16}{(2n-1)\pi}\right]\cos(2n -\right.$$
$$+ \frac{4}{(2n-1)^2}\sin$$

45. $$u(x,t) = 6 - \frac{6x}{\pi} + \sum_{n=1}^{\infty}\frac{-6}{n\pi}\cos 8nt \sin 2nx + \left[\left(1 - \frac{12}{5\pi}\right)\cos 20t + \frac{1}{21\pi}\sin 20t\right]\sin$$
$$+ \sum_{n=1}^{2}\left\{\frac{-12}{(2n-1)\pi}\cos 4(2n-1)t + \frac{1}{\pi[(2n-1)^2-4]}\sin 4(2n-1)t\right\}\sin($$
$$+ \sum_{n=4}^{\infty}\left\{\frac{-12}{(2n-1)\pi}\cos 4(2n-1)t + \frac{1}{\pi[(2n-1)^2-4]}\sin 4(2n-1)t\right\}\sin($$

47. The initial shape of the string is that of the curve $\sin 3x$ with $0 < x < \pi$, a released with an initial velocity of 4 units per second.

49. (e) u_1 and u_2 are damped waves travelling to the right with speed $\beta/\sqrt{\gamma}$.

51. $\theta(x, t) = \sum_{n=1}^{\infty} \frac{24(-1)^{n+1}}{(2n-1)^2\pi^2} \cos \frac{(2n-1)\pi t}{2} \sin \frac{(2n-1)\pi x}{2}$.

53. $u(x, t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2\pi^2} \cos (2n-1)\pi t \sin (2n-1)\pi x$.

55. $u(x, t) = \sum_{n=1}^{\infty} \frac{72}{n^2\pi^2} \sin \frac{n\pi}{3} \sin \frac{n\pi}{6} \sin \frac{n\pi t}{6} \sin \frac{n\pi x}{6}$.

57. $u(x, t) = \sum_{n=1}^{\infty} \frac{120}{n^2\pi^2} \sin \frac{7n\pi}{20} \sin \frac{n\pi}{20} \sin \frac{n\pi t}{10} \sin \frac{n\pi x}{10}$.

59. $\rho S T_n'' + \left[EI \frac{n^4\pi^4}{l^4} + \frac{n^2\pi^2}{l^2} F(t)\right] T_n = 0$.

63. 3.932 Newtons/m.

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1. $u(x, t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} e^{-n^2\pi^2 t} \sin n\pi x$.

3. $u(x, t) = \sum_{n=1}^{\infty} \frac{-\pi}{n} e^{-16n^2 t} \sin 2nx$
$+ \sum_{n=1}^{\infty} \left[\frac{-2\pi}{2n-1} - \frac{8}{\pi(2n-1)^3}\right] e^{-4(2n-1)^2 t} \sin (2n-1)x$.

5. $u(x, t) = \sum_{n=1}^{\infty} \left\{\frac{2n\pi}{n^2\pi^2 - 1}[(-1)^{n-1}\cos 1 + 1] + \frac{6(-1)^{n+1}}{n\pi}\right\} e^{-2n^2\pi^2 t} \sin n\pi x$.

7. $u(x, t) = \sum_{n=1}^{\infty} \frac{2n\pi}{n^2\pi^2 + 1}[1 + e(-1)^{n+1}]\, e^{-5n^2\pi^2 t} \sin n\pi x$.

9. $u(x, t) = 3e^{-t} \sin x + \sum_{n=2}^{\infty} \frac{2(-1)^{n+1}}{n} e^{-n^2 t} \sin nx$.

11. $u(x,t) = \sum_{n=1}^{\infty} \frac{4}{\pi^3(2n-1)^3} e^{-(2n-1)^2\pi^2 t} \sin (2n-1)\pi x.$

13. $u(x,t) = \sum_{n=1}^{\infty} \frac{3}{2n^3} e^{-8n^2 t} \sin 2nx$

$+ \sum_{n=1}^{\infty} \frac{4}{(2n-1)^3} [2\pi^2(2n-1)^2 + 1] e^{-2(2n-1)^2 t} \sin (2n-1)x.$

15. $u(x,t) = \sum_{n=1}^{\infty} 2\left\{\frac{(-1)^{n+1}}{n\pi} + \frac{n\pi}{n^2\pi^2+1}[1 + e(-1)^{n+1}]\right\} e^{-n^2\pi^2 t} \sin n\pi x$

17. $u(x,t) = \frac{10(\pi - x)}{\pi} + \sum_{n=1}^{\infty} \frac{2}{n\pi}[10 + (-1)^{n-1}\pi] e^{-5n^2 t} \sin nx.$

19. $u(x,t) = 7 - 4x + \sum_{n=1}^{\infty} \left\{\frac{2}{n\pi}[3(-1)^n - 7] + \frac{2}{n^3\pi^3}[1 - (-1)^n]\right\} e^{-n}$

21. $u(x,t) = 7 - 5x + \sum_{n=1}^{\infty} \left\{\frac{-10}{n\pi} + \frac{4n\pi}{n^2\pi^2 - 4}[1 - (-1)^n \cos 2]\right\} e^{-n^2\pi^2}$

23. $u(x,t) = 7 - 4x + \sum_{n=1}^{\infty} \left\{\frac{2n\pi}{n^2\pi^2+1}[1 + e(-1)^{n+1}] + \frac{2}{n\pi}[3(-1)^n -\right.$

25. $u(x,t) = 10 + \sum_{n=1}^{\infty} \left\{\frac{4}{n^3\pi^3}[(-1)^n + 2] + \frac{20}{n\pi}[(-1)^n - 1]\right\} e^{-5n^2\pi^2 t}$ si

27. $u(x,t) = e^{-\alpha t} \sum_{n=1}^{\infty} \frac{4}{(2n-1)^3\pi^3} e^{-(2n-1)^2\pi^2 t} \sin (2n-1)\pi x.$

29. $u(x,t) = \sum_{n=1}^{\infty} \frac{2A}{n\pi}\left[1 - \cos\frac{n\pi}{2}\right] e^{-(n^2\pi^2 at)/l^2} \sin \frac{n\pi x}{l}.$

31. $u(15,t) = \sum_{n=1}^{\infty} \frac{200}{n\pi}\left[\cos\frac{2n\pi}{3} - (-1)^n\right] \sin\frac{n\pi}{2} e^{-[n^2\pi^2(1.02)t]/30^2}.$

33. (a) $v(r,t) = \sum_{n=1}^{\infty} c_n e^{-(n^2\pi^2 at)/R^2} \sin\frac{n\pi r}{R}$, with $c_n = \frac{2}{R}\int_0^R rg(r) \sin\frac{n\pi}{R}$

(b) $u(r,t) = \frac{1}{r} v(r,t)$ with v given in part (a). Note that $\lim_{r\to 0+} u($

35. (a) $v(r,t) = \sum_{n=1}^{\infty} \frac{12(-1)^{n+1}R^3}{n^3\pi^3} e^{-(n^2\pi^2 at)/R^2} \sin\frac{n\pi r}{R}$.

(b) $u(r,t) = \frac{1}{r}v(r,t)$ with v given in part (a). Note that $\lim_{r\to 0+} u(r,t)$ makes sense.

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1. $u(x,y) = \sum_{n=1}^{\infty}\left[\frac{2(-1)^{n+1}}{n\pi}\cosh n\pi y + \frac{2(-1)^n}{n\pi}\coth n\pi \sinh n\pi y\right]\sin n\pi x.$

3. $u(x,y) = \sum_{n=1}^{\infty} 2\left[\frac{2-n^2\pi^2}{n^3\pi^3}(-1)^n - \frac{2}{n^3\pi^3}\right][\cosh n\pi y - \coth n\pi \sinh n\pi y]\sin n\pi x.$

5. $u(x,y) = [\cosh \pi y - \coth \pi \sinh \pi y]\sin \pi x.$

7. $u(x,y) = \sum_{n=1}^{\infty}\frac{2n\pi}{(\sinh n\pi)(n^2\pi^2-1)}[1+(-1)^{n+1}\cos 1]\sinh n\pi y \sin n\pi x.$

9. $u(x,y) = \sum_{n=1}^{\infty}\frac{2n\pi}{(n^2\pi^2+4)\sinh(n\pi/2)}[1-(-1)^n e^2]\sinh\frac{n\pi y}{2}\sin\frac{n\pi x}{2}.$

11. $$u(x,y) = \left[\frac{-2}{\pi}\cosh \pi y + \left(1 - \frac{\cosh\pi}{\pi}\right)\frac{\sinh \pi y}{\sinh \pi}\right]\sin \pi x + \sum_{n=2}^{\infty}\frac{2(-1)^{n-1}}{n\pi}[\cosh n\pi y - \coth n\pi \sinh n\pi y]\sin n\pi x.$$

13. $$u(x,y) = \sum_{n=1}^{\infty}\left\{\frac{2n\pi}{n^2\pi^2+4}[1-(-1)^n e^2]\cosh\frac{n\pi y}{2} + \left[\frac{2n\pi(1-(-1)^n\cos 2)}{(n^2\pi^2-4)\sinh n\pi} - \frac{2n\pi(\coth n\pi)(1-(-1)^n e^2)}{n^2\pi^2+4}\right]\cdot\sinh\frac{n\pi y}{2}\right\}\sin\frac{n\pi x}{2}.$$

15. $$u(x,y) = \sum_{n=1}^{\infty}\left\{2\left[\frac{2-n^2\pi^2}{n^3\pi^3}(-1)^n - \frac{2}{n^3\pi^3}\right]\cosh n\pi y + \left[\frac{2n\pi(1+(-1)^{n+1}e)}{(n^2\pi^2+1)\sinh 2n\pi} - 2\coth 2n\pi\left(\frac{2-n^2\pi^2}{n^3\pi^3}(-1)^n - \frac{2}{n^3\pi^3}\right)\right]\cdot\sinh n\pi y\right\}\sin n\pi x.$$

17. $u(x, y) = \sum_{n=1}^{\infty} \left\{ \frac{(-1)^{n+1} 2n\pi \sin 1}{n^2\pi^2 - 1} [\cosh n\pi x - \coth n\pi \sinh n\pi x] \right.$

19. $u(x, y) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi \sinh n\pi} \sinh n\pi x \sin n\pi y.$

21. $u(x, y) = \sum_{n=1}^{\infty} \frac{2n\pi[1 + (-1)^{n+1} \cos 1]}{n^2\pi^2 - 1} [\cosh n\pi x - \coth n\pi \sinh$

23. $u(x, y) = \sum_{n=1}^{\infty} \frac{-8[2 + (-1)^n(n^2\pi^2 - 2)]}{n^3\pi^3 \sinh \frac{n\pi}{2}} \sinh \frac{n\pi x}{2} \sin \frac{n\pi y}{2}.$

25. $u(x, y) = \sum_{n=1}^{\infty} \frac{2n\pi[1 + (-1)^{n+1} e]}{n^2\pi^2 + 1} [\cosh n\pi x - \coth 2n\pi \sinh n\pi$

27. $u(x, y) = \sum_{n=1}^{\infty} \left\{ \frac{2n\pi[1 + (-1)^{n+1} e^3]}{n^2\pi^2 + 9} \cosh n\pi x \right.$

$+ \left[\frac{6(-1)^{n+1}}{n\pi \sinh n\pi} - \frac{2n\pi(\coth n\pi)(1 + (-1)^{n+1} e^3)}{n^2\pi^2 + 9} \right] \sin$

29. $u(x, y) = \sum_{n=1}^{\infty} \frac{4}{n\pi} [1 - 2(-1)^n] \left[\cosh \frac{n\pi x}{2} - \coth n\pi \sinh \frac{n\pi x}{2} \right]$ s

31. $u(x, y) = \sum_{n=1}^{\infty} \frac{4}{(2n - 1)\pi} \left[\cosh \frac{(2n - 1)\pi x}{2} \right.$

$+ \frac{3 - \cosh \frac{(2n - 1)3\pi}{2}}{\sinh \frac{(2n - 1)3\pi}{2}} \sinh \frac{(2n - 1)\pi x}{2} \Bigg] \sin \frac{(2n -}{2}$

33. $u(x, y) = u_1(x, y) + u_2(x, y)$, where u_1 is the solution in Exe solution in Exercise 17.

35. $u(x, y) = u_1(x, y) + u_2(x, y)$, where u_1 is the solution in Exe solution in Exercise 25.

37. $u(x, y) = u_1(x, y) + u_2(x, y)$, where u_1 is the solution in Exe

$u_2(x, y) = \left[\cosh \pi y + \left(\frac{2}{\pi} - \frac{8}{\pi^3} - \cosh \pi \right) \frac{\sinh \pi y}{\sinh \pi} \right] \sin \pi x$

$+ \sum_{n=2}^{\infty} \left[\frac{2 - n^2\pi^2}{n^3\pi^3} (-1)^n - \frac{2}{n^3\pi^3} \right] \frac{2 \sinh n\pi y}{\sinh n\pi} \sin n\pi x.$

39. $u(x, y) = u_1(x, y) + u_2(x, y)$, where u_1 is the solution in Exercise 13, and

$$u_2(x, y) = \sum_{n=1}^{\infty} \left\{ \frac{2n\pi[1 - (-1)^n e^2]}{n^2\pi^2 + 4} \cosh \frac{n\pi x}{2} \right.$$

$$+ \left[\frac{2n\pi(1 - (-1)^n \cos 2)}{(n^2\pi^2 - 4) \sinh 2n\pi} - \frac{2n\pi \coth 2n\pi \, (1 - (-1)^n e^2)}{n^2\pi^2 + 4} \right]$$

$$\left. \cdot \sinh \frac{n\pi x}{2} \right\} \sin \frac{n\pi y}{2} .$$

41. $$u(x, y) = \sum_{n=1}^{\infty} \frac{200}{n\pi} [1 - (-1)^n] \left[\cosh \frac{n\pi y}{10} - \coth \frac{3n\pi}{2} \sinh \frac{n\pi y}{10} \right] \sin \frac{n\pi x}{10} .$$

43. $u(r, \theta) = \sum_{n=0}^{\infty} r^n (\alpha_n \cos n\theta + \beta_n \sin n\theta)$, with $\alpha_n = \dfrac{1}{a^n \pi} \displaystyle\int_0^{2\pi} f(\theta) \cos n\theta \, d\theta$,

$$\beta_0 = 0, \quad \beta_n = \frac{1}{a^n \pi} \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta .$$

45. $$u(r, \theta) = \sum_{n=1}^{\infty} \frac{200}{n\pi} [1 - (-1)^n] r^n \sin n\theta .$$

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1. $$u(x, t) = \sum_{n=1}^{\infty} \frac{2}{n^5\pi^5} \{ [e^{-n^2\pi^2 t} - 1] [1 - (-1)^n + (-1)^n n^2\pi^2]$$

$$+ (-1)^{n+1} n^4\pi^4 e^{-n^2\pi^2 t} - [(-1)^n - 1] n^2\pi^2 t \} \sin n\pi x .$$

3. $$u(x, t) = \frac{2 \sin x}{\pi} (\sin t + \cos t)$$

$$+ \sum_{n=1}^{\infty} \frac{4 \sin (2n + 1)x}{\pi(2n + 1) [(2n + 1)^4 + 1]} [\sin t + (2n + 1)^2 \cos t - (2n + 1)^2 e^{-(2n+1)^2 t}] .$$

5. $$u(x, t) = \sum_{n=1}^{\infty} \left\{ \frac{2[2(-1)^n - 3]}{5 n^3\pi^3} (1 - e^{5n^2\pi^2 t}) \right.$$

$$\left. + \frac{2n\pi}{n^2\pi^2 + 1} [1 + e(-1)^{n-1}] \right\} e^{-5n^2\pi^2 t} \sin n\pi x .$$

7. $$u(x, t) = \left\{\frac{1}{5}[-3 + (3\cos t + \sin t)e^{3t} - \pi(e^{3t} - 1)\right\}\frac{2e^{-3t}\sin x}{\pi}$$

$$+ \frac{1}{4}(1 - e^{-12t})\sin 2x + \left\{\frac{1}{1095}[-27 + (27\cos t + \sin t)e^{27t}] - \frac{\pi}{27}(e^{27t} -\right.$$

$$\cdot \frac{2e^{-27t}\sin 3x}{\pi} + \sum_{n=4}^{\infty}\left\{\frac{1 - (-1)^n}{n(9n^4 + 1)}[-3n^2 + (3n^2\cos t + \sin t)e^{3n^2t}] + \frac{(-1)^n}{n^3}\right.$$

$$\cdot \frac{2e^{-3n^2t}\sin nx}{\pi}.$$

9. $$u(x, t) = \sum_{n=1}^{\infty}\left\{\frac{(-1)^{n+1}t^2e^{2n^2t}}{2n^2} + \frac{(2n^2t - 1)(-1)^n e^{2n^2t}}{4n^6} + \frac{(-1)^n}{4n^6} + 2\pi^2[1 -\right.$$

$$\left. + \frac{1}{n^2}[4 + 2(-1)^n]\right\}\frac{e^{-2n^2t}\sin nx}{n}.$$

11. $$u(x, t) = \sum_{n=1}^{\infty}\left\{\left[\frac{2 - 3(-1)^n}{n^3\pi^3}\right]\cos n\pi t - \left[\frac{1 - (-1)^n}{n^4\pi^4}\right]\sin n\pi t\right.$$

$$\left. + \frac{(-1)^n}{n^3\pi^3} - \left[\frac{(-1)^n - 1}{n^3\pi^3}\right]t\right\}2\sin n\pi x.$$

13. $$u(x, t) = \sum_{n=1}^{\infty}\left\{\frac{-\cos(2n - 1)t}{(2n - 1)[(2n - 1)^2 - \pi^2]} + \frac{3\sin(2n - 1)t}{(2n - 1)^2}\right.$$

$$\left. + \frac{\cos \pi t}{(2n - 1)[(2n - 1)^2 - \pi^2]}\right\}\frac{4\sin(2n - 1)x}{\pi}.$$

15. $$u(x, t) = \sum_{n=1}^{\infty}\left\{\left[\frac{7}{n^3\pi^3}\right]\cos n\pi t + \left[\frac{1 - (-1)^n}{n^2\pi}\right]\sin n\pi t\right.$$

$$\left. - \frac{\sqrt{2}}{n^3\pi^3}[3 - 2(-1)^n]\right\}2\sin n\pi x.$$

17. $$u(x, t) = \sum_{n=1}^{\infty}\left\{\left[\frac{-1 + (-1)^n}{n(n^2 - \pi^2)} - \frac{3(-1)^n\pi}{n^3}\right]\cos nt + \frac{\pi}{n^2}[1 - (-1)^n]\sin nt\right.$$

$$\left. - \left[\frac{(-1)^n - 1}{n(n^2 - \pi^2)}\right]\cos \pi t + \frac{3\pi(-1)^n}{n^3}\right\}\frac{2\sin nx}{\pi}.$$

19. $$u(x, t) = \sum_{n=1}^{\infty}\left\{\left[\frac{2\pi^3}{n}(1 - (-1)^n) + \frac{\pi}{n^3}(4 + 2(-1)^n) - \frac{2\pi(-1)^n}{n^5}\right]\cos nt\right.$$

$$\left. - \frac{\pi(-1)^n t^2}{n^3} + \frac{2\pi(-1)^n}{n^5}\right\}\frac{2\sin nx}{\pi}.$$

21. $u(x, t) = \sum_{n=1}^{\infty} T_n(t)\phi_n(x)$, where $\phi_n(x) = \sqrt{2/l}\sin\frac{(2n-1)\pi x}{2l}$ and $T_n(t)$ is the solution of the initial value problem

$$T_n'' + c^2\mu_n T_n = K_n(t),$$

$$T_n(0) = \int_0^l f(x)\phi_n(x)dx, \quad T_n'(0) = \int_0^l g(x)\phi_n(x)dx,$$

with

$$K_n(t) = \int_0^l F(x, t)\phi_n(x)dx,$$

and

$$\mu_n = \frac{(2n-1)^2\pi^2}{4l^2}.$$

23. $u(x, t) = \sum_{n=1}^{\infty} \frac{[1-(-1)^n]l}{n^3\pi^3}\left\{gl\left(\cos\frac{n\pi t}{l} - 1\right) + 3n\pi \sin\frac{n\pi t}{l}\right\}\sin\frac{n\pi x}{l}.$

25. $u(x, t) = (1 - \cos \pi t) \sin \pi x.$

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1. $u(x, t) = (1 - x)t^2 + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} e^{-n^2\pi^2 t} \sin n\pi x.$

3. $u(x, t) = \frac{1}{2}(1 - x)t^2 + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}e^{-n^2\pi^2 t}}{n^5\pi^5}[1 + n^4\pi^4 - (1 - n^2\pi^2 t)e^{n^2\pi^2 t}] \cdot \sin n\pi x.$

5. $u(x, t) = (1 - x)2t + \sum_{n=1}^{\infty} \left\{\frac{5(-1)^{n-1}}{3 + n^2\pi^2}[-1 + e^{(3-n^2\pi^2)t}] + \frac{2}{n^2\pi^2}[1 - e^{n^2\pi^2 t}] + (-1)^{n-1}\right\}\frac{2e^{-n^2\pi^2 t}\sin n\pi x}{n\pi}.$

7. $u(x, y) = \frac{1}{12}x(x^3 - 1) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{6n\pi}[\cosh n\pi y - \coth n\pi \sinh n\pi y] \sin n\pi x.$