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ORDINARY DIFFERENTIAL EQUATIONS
Chapter 10: Fourier Series
Student Solution Manual

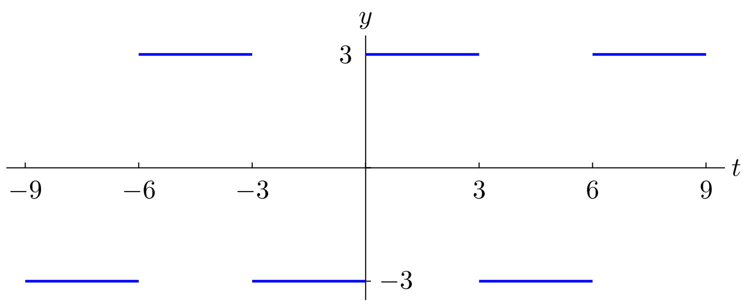
November 11, 2015

Springer

Chapter 1
Solutions

Section 10.1

1.



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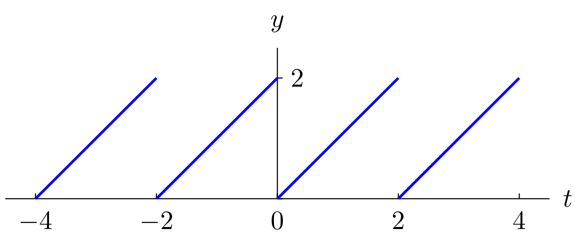

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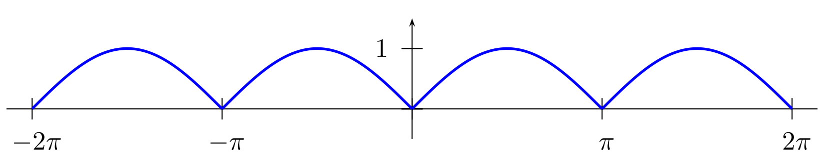

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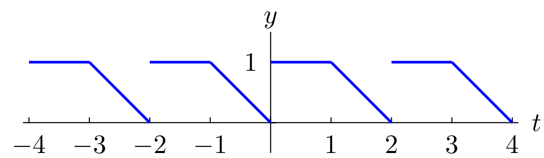

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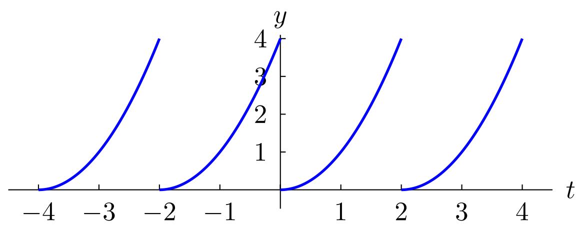


11. Periodic. Fundamental period is $2\pi/2 = \pi$.

13. Since $\cos 2t$ is periodic with fundamental period $2\pi/2 = \pi$, it follows that all positive multiples $k\pi$ is also a period. Similarly, $\sin 3t$ is periodic with fundamental period $2\pi/3$ so that all positive multiples $2m\pi/3$ are also periods. If p is any number that can be written both as $k\pi$ and $2m\pi/3$ for appropriate k and m, then p is a period for the sum: $\cos 2(t+p)+\sin 3(t+p) = \cos(2t+2p)+\sin(3t+3(2m\pi/3)) = \cos(2t+2k\pi)+\sin(3t+2m\pi) = \cos 2t + \sin 3t$. Therefore, the function is periodic with period p. The smallest p that is both $k\pi$ and $2m\pi/3$ is $p = 2\pi$ ($k = 2$, $m = 3$). Thus the fundamental period is 2π

15. $\sin^2 t = (1 - \cos 2t)/2$ so $\sin^2 t$ is periodic with fundamental period $2\pi/2 = \pi$

17. Periodic. The periods of $\sin t$ are $2k\pi$, the periods of $\sin 2t$ are $m\pi$, and the periods of $\sin 3t$ are $2n\pi/3$ for positive integers k, m, n. The smallest p that is common to all of these is $p = 2\pi$, so the fundamental period is 2π.

19. $f(-t) = (-t)\,|-t| = -t\,|t| = -f(t)$ for all t. Thus, $f(t)$ is odd.

21. This is the product of two even functions ($\cos t$ for both). Thus it is even by Proposition 5 (1).

23. $f(-t) = f(t) \implies (-t)^2 + \sin(-t) = t^2 + \sin t \implies t^2 - \sin t = t^2 + \sin t \implies 2\sin t = 0 \implies t = k\pi$. Thus $f(t)$ is not even. Similarly, $f(t)$ is not odd.

25. $f(-t) = \ln|\cos(-t)| = \ln|\cos t| = f(t)$. Thus, $f(t)$ is even.

27. Use the identity $\cos A \sin B = \frac{1}{2}(\sin(A+B) + \sin(B-A))$ to get

$$\int_{-L}^{L} \cos\frac{n\pi}{L}t \sin\frac{m\pi}{L}t\,dt = \frac{1}{2}\int_{-L}^{L}\big(\sin\frac{(m+n)\pi}{L}t + \sin\frac{(m-n)\pi}{L}t\big)\,dt$$
$$= \frac{1}{2}\left(\frac{-L}{(m+n)\pi}\cos\frac{(m+n)\pi}{L}t + \frac{-L}{(m-n)\pi}\cos\frac{(m-n)\pi}{L}t\right)\Bigg|_{-L}^{L} = 0.$$

Section 10.2

1. The period is 10 so $2L = 10$ and $L = 5$. Then

$$a_0 = \frac{1}{5}\int_{-5}^{5} f(t)\,dt = \frac{1}{5}\int_{-5}^{0} 0\,dt + \frac{1}{5}\int_{0}^{5} 3\,dt = \frac{1}{5}\cdot 15 = 3.$$

For $n \geq 1$,

$$\begin{aligned}
a_n &= \frac{1}{5}\int_{-5}^{5} f(t)\cos\frac{n\pi}{5}t\,dt = \frac{1}{5}\int_{-5}^{0} f(t)\cos\frac{n\pi}{5}t\,dt + \frac{1}{5}\int_{0}^{5} f(t)\cos\frac{n\pi}{5}t\,dt \\
&= \frac{1}{5}\int_{-5}^{0}(0)\cos\frac{n\pi}{5}t\,dt + \frac{1}{5}\int_{0}^{5} 3\cos\frac{n\pi}{5}t\,dt \\
&= \frac{1}{5}\left[\frac{15}{n\pi}\sin\frac{n\pi}{5}t\right]_0^5 = 0,
\end{aligned}$$

and

$$\begin{aligned}
b_n &= \frac{1}{5}\int_{-5}^{5} f(t)\sin\frac{n\pi}{5}t\,dt = \frac{1}{5}\int_{-5}^{0} f(t)\sin\frac{n\pi}{5}t\,dt + \frac{1}{5}\int_{0}^{5} f(t)\sin\frac{n\pi}{5}t\,dt \\
&= \frac{1}{5}\int_{-5}^{0}(0)\sin\frac{n\pi}{5}t\,dt + \frac{1}{5}\int_{0}^{5} 3\sin\frac{n\pi}{5}t\,dt \\
&= \frac{1}{5}\left[-\frac{15}{n\pi}\cos\frac{n\pi}{5}t\right]_0^5 \\
&= -\frac{3}{n\pi}(\cos n\pi - 1) = \frac{3}{n\pi}(1-(-1)^n) \\
&= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{6}{n\pi} & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

Therefore, the Fourier series is

$$f(t) \sim \frac{3}{2} + \frac{6}{\pi}\left(\sin\frac{\pi}{5}t + \frac{1}{3}\sin\frac{3\pi}{5}t + \frac{1}{5}\sin\frac{5\pi}{5}t + \frac{1}{7}\sin\frac{7\pi}{5}t + \cdots\right).$$
$$= \frac{3}{2} + \frac{6}{\pi}\sum_{n=\text{odd}}\frac{1}{n}\sin\frac{n\pi}{5}t.$$

3. The period is 2π so $L = \pi$. Then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_{-\pi}^{0} 4\,dt + \frac{1}{\pi}\int_{0}^{\pi} -1\,dt = 4 - 1 = 3.$$

For $n \geq 1$,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt = \frac{1}{\pi}\int_{-\pi}^{0} f(t)\cos nt\,dt + \frac{1}{\pi}\int_{0}^{\pi} f(t)\cos nt\,dt \\ &= \frac{1}{\pi}\int_{-\pi}^{0} 4\cos nt\,dt + \frac{1}{\pi}\int_{0}^{\pi}(-1)\cos nt\,dt \\ &= \frac{1}{\pi}\left[\frac{4}{n}\sin nt\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{-1}{n}\sin nt\right]_{0}^{\pi} = 0, \end{aligned}$$

and

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt = \frac{1}{\pi}\int_{-\pi}^{0} f(t)\sin nt\,dt + \frac{1}{\pi}\int_{0}^{\pi} f(t)\sin nt\,dt \\ &= \frac{1}{\pi}\int_{-\pi}^{0} 4\sin nt\,dt + \frac{1}{\pi}\int_{0}^{\pi}(-1)\sin nt\,dt \\ &= \frac{1}{\pi}\left[-\frac{4}{n}\cos nt\right]_{-\pi}^{0} + \frac{1}{\pi}\left[-\frac{-1}{n}\cos nt\right]_{0}^{\pi} \\ &= \frac{-4}{n\pi}(1 - \cos(-n\pi)) + \frac{1}{n\pi}(\cos(n\pi) - 1) \\ &= -\frac{5}{n\pi}(1 - \cos n\pi) = -\frac{5}{n\pi}(1 - (-1)^n). \end{aligned}$$

Therefore,

$$b_n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ -\frac{10}{n\pi} & \text{if } n \text{ is odd,} \end{cases}$$

and the Fourier series is

$$f(t) \sim \frac{3}{2} - \frac{10}{\pi}\left(\sin nt + \frac{1}{3}\sin nt + \frac{1}{5}\sin nt + \frac{1}{7}\sin nt + \cdots\right)$$
$$= \frac{3}{2} - \frac{10}{\pi}\sum_{n=\text{odd}}\frac{1}{n}\sin nt.$$

5. The period is 2π so $L = \pi$. The function $f(t)$ is odd, so the cosine terms a_n are all 0. Now compute the coefficients b_n:

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt \\
&= \frac{2}{\pi}\int_{0}^{\pi} t\sin nt\,dt \qquad \left(\text{let } x = nt \text{ so } t = \frac{1}{n}x \text{ and } dt = \frac{1}{n}dx\right) \\
&= \frac{2}{\pi}\int_{0}^{n\pi} \frac{1}{n}x\sin x\,\frac{1}{n}\,dx = \frac{2}{n^2\pi}\int_{0}^{n\pi} x\sin x\,dx \\
&= \frac{2}{n^2\pi}\left[\sin x - x\cos x\right]_{x=0}^{x=n\pi} \\
&= -\frac{2}{n^2\pi}(n\pi\cos n\pi) = -\frac{2}{n}(-1)^n.
\end{aligned}$$

Therefore, the Fourier series is

$$\begin{aligned}
f(t) &\sim 2\left(\sin t - \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t - \frac{1}{4}\sin 4t + \cdots\right) \\
&= 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin nt.
\end{aligned}$$

7. The period is 4 so $L = 2$. The function is even, so the sine terms $b_n = 0$. For the cosine terms a_n:

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(t)\,dt = \frac{1}{2}2\int_{0}^{2} f(t)\,dt = \int_{0}^{2} t^2\,dt = \left.\frac{t^3}{3}\right|_0^2 = \frac{8}{3},$$

and for $n \geq 1$, (integration by parts is used multiple times)

$$\begin{aligned}
a_n &= \frac{1}{2}\int_{-2}^{2} f(t)\cos\frac{n\pi}{2}t\,dt = \int_{0}^{2} f(t)\cos\frac{n\pi}{2}t\,dt = \int_{0}^{2} t^2\cos\frac{n\pi}{2}t\,dt \\
&= t^2\cdot\frac{2}{n\pi}\sin\left.\frac{n\pi}{2}t\right|_0^2 - \int_0^2 \frac{4t}{n\pi}\sin\frac{n\pi}{2}t\,dt = -\frac{4}{n\pi}\int_0^2 t\sin\frac{n\pi}{2}t\,dt \\
&= -\frac{4}{n\pi}\left[\frac{-2t}{n\pi}\cos\left.\frac{n\pi}{2}t\right|_0^2 + \frac{2}{n\pi}\int_0^2 \cos\frac{n\pi}{2}t\,dt\right] \\
&= \frac{16}{n^2\pi^2}\cos n\pi - \frac{16}{n^3\pi^3}\sin\left.\frac{n\pi}{2}t\right|_0^2 \\
&= \frac{16}{n^2\pi^2}(-1)^n.
\end{aligned}$$

Therefore, the Fourier series is

$$f(t) \sim \frac{4}{3} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi}{2} t.$$

9. The period is π so $L = \pi/2$ and $n\pi/L = 2n$. The function is even, so the sine terms $b_n = 0$. For the cosine terms a_n:

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(t)\, dt = \frac{2}{\pi} \int_0^{\pi} \sin t\, dt = -\frac{2}{\pi} \cos t \Big|_0^{\pi} = \frac{4}{\pi},$$

and for $n \geq 1$,

$$\begin{aligned}
a_n &= \frac{2}{\pi} \int_0^{\pi} f(t) \cos 2nt\, dt = \frac{2}{\pi} \int_0^{\pi} \sin t \cos 2nt\, dt \\
&= \frac{2}{\pi} \int_0^{\pi} \frac{1}{2} (\sin(2n+1)t - \sin(2n-1)t)\, dt \\
&= \frac{1}{\pi} \left[\frac{-1}{2n+1} \cos(2n+1)t + \frac{1}{2n-1} \cos(2n-1)t \right]_0^{\pi} \\
&= \frac{1}{\pi} \left[\frac{-1}{2n+1} (\cos(2n+1)\pi - 1) + \frac{1}{2n-1} (\cos(2n-1)\pi - 1) \right] \\
&= \frac{-2}{\pi} \left[\frac{1}{2n-1} - \frac{1}{2n+1} \right] = \frac{-4}{(4n^2-1)\pi}.
\end{aligned}$$

Therefore, the Fourier series is

$$f(t) \sim \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nt}{4n^2 - 1}.$$

11. The period is 2 so $L = 1$. Since the function $f(t)$ is even, the sine coefficients $b_n = 0$. Now compute the coefficients a_n: For $n = 0$, using the fact that $f(t)$ is even,

$$\begin{aligned}
a_0 &= \int_{-1}^{1} f(t)\, dt = 2 \int_0^1 f(t)\, dt \\
&= 2 \int_0^1 (1 - t)\, dt = 2 \left[t - \frac{t^2}{2} \right]_0^1 = 1.
\end{aligned}$$

For $n \geq 1$, using the fact that $f(t)$ is even,

$$\begin{aligned}
a_n &= \int_{-1}^{1} f(t)\cos n\pi t\,dt = 2\int_0^1 f(t)\cos n\pi t\,dt \\
&= 2\int_0^1 (1-t)\cos n\pi t\,dt \qquad \text{(integration by parts with } u = 1-t,\ dv = \cos n\pi t\,dt) \\
&= 2\left[\frac{1-t}{n\pi}\sin n\pi t\right]_0^1 + \frac{2}{n\pi}\int_0^1 \sin n\pi t\,dt \\
&= -\frac{2}{n^2\pi^2}\cos n\pi t\Big|_0^1 \\
&= -\frac{2}{n^2\pi^2}[\cos n\pi - 1] = -\frac{2}{n^2\pi^2}[(-1)^n - 1]
\end{aligned}$$

Therefore,

$$a_n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{4}{n^2\pi^2} & \text{if } n \text{ is odd} \end{cases}$$

and the Fourier series is

$$\begin{aligned}
f(t) &\sim \frac{1}{2} + \frac{4}{\pi^2}\left(\frac{\cos\pi t}{1^2} + \frac{\cos 3\pi t}{3^2} + \frac{\cos 5\pi t}{5^2} + \frac{\cos 7\pi t}{7^2} + \cdots\right) \\
&= \frac{1}{2} + \frac{4}{\pi^2}\sum_{n=\text{odd}}^{\infty}\frac{\cos n\pi t}{n^2}.
\end{aligned}$$

13. The period is 2π so $L = \pi$. The function $f(t)$ is an odd function, so the cosine terms $a_n = 0$. Now compute the coefficients b_n: Since $f(t)$ is odd, $f(t)\sin nt$ is even so, (using integration by parts multiple times)

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt = \frac{2}{\pi}\int_0^{\pi} f(t)\sin nt\,dt \\
&= \frac{2}{\pi}\int_0^{\pi} t(\pi - t)\sin nt\,dt \\
&= \frac{2}{\pi}\,\frac{-t(\pi - t)}{n}\cos nt\Big|_0^{\pi} + \frac{2}{n\pi}\int_0^{\pi}(\pi - 2t)\cos nt\,dt \\
&= \frac{2(\pi - 2t)}{n^2\pi}\sin nt\Big|_0^{\pi} + \frac{4}{n^2\pi}\int_0^{\pi}\sin nt\,dt \\
&= -\frac{4}{n^3\pi}\cos nt\Big|_0^{\pi} = -\frac{4}{n^3\pi}(\cos n\pi - 1) \\
&= -\frac{4}{n^3\pi}((-1)^n - 1) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{8}{n^3\pi} & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

Therefore the Fourier series is

$$f(t) \sim \frac{8}{\pi} \sum_{n=\mathrm{odd}} \frac{\sin nt}{n^3}.$$

15. The function is odd of period 2π so the cosine terms $a_n = 0$. Let $n \geq 1$. Then,

$$\begin{aligned}
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt\, dt = \frac{2}{\pi} \int_0^{\pi} f(t) \sin nt\, dt \\
&= \frac{2}{\pi} \int_0^{\pi} \sin \frac{t}{2} \sin nt\, dt \\
&= \frac{1}{\pi} \int_0^{\pi} (\cos(\frac{1}{2} - n)t - \cos(\frac{1}{2} + n)t)\, dt \\
&= \frac{1}{\pi} \left[\frac{\sin(\frac{1}{2} - n)t}{\frac{1}{2} - n} - \frac{\sin(\frac{1}{2} + n)t}{\frac{1}{2} + n} \right]_0^{\pi} \\
&= \frac{1}{\pi} \left[\frac{\sin(\frac{1}{2} - n)\pi}{\frac{1}{2} - n} - \frac{\sin(\frac{1}{2} + n)\pi}{\frac{1}{2} + n} \right] \\
&= \frac{1}{\pi} \left[\frac{\sin \frac{\pi}{2} \cos n\pi}{\frac{1}{2} - n} - \frac{\sin \frac{\pi}{2} \cos n\pi}{\frac{1}{2} + n} \right] \\
&= \frac{(-1)^n}{\pi} \left[\frac{1}{\frac{1}{2} - n} - \frac{1}{\frac{1}{2} + n} \right] \\
&= \frac{(-1)^n}{\pi} \left[\frac{(\frac{1}{2} + n) - (\frac{1}{2} - n)}{\frac{1}{4} - n^2} \right] \\
&= \frac{2n(-1)^{n+1}}{\pi(n^2 - \frac{1}{4})}.
\end{aligned}$$

Therefore, the Fourier series is

$$f(t) \sim \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2 - \frac{1}{4}} \sin nt.$$

17. The period is 2 so $L = 1$.

$$a_0 = \int_{-1}^{1} e^t\, dt = e^1 - e^{-1} = 2 \sinh 1.$$

For $n \geq 1$, the following integration formulas (with $a = 1$, $b = n\pi$) will be useful.

$$\int e^{at}\cos(bt)\,dt = \frac{1}{a^2+b^2}e^{at}[a\cos(bt)+b\sin(bt)]+C$$
$$\int e^{at}\sin(bt)\,dt = \frac{1}{a^2+b^2}e^{at}[a\sin(bt)-b\cos(bt)]+C$$

Then,

$$\begin{aligned} a_n &= \int_{-1}^{1} e^t \cos n\pi t\,dt \\ &= \frac{1}{1+n^2\pi^2}e^t[\cos n\pi t + n\pi \sin n\pi t]\Big|_{-1}^{1} \\ &= \frac{1}{1+n^2\pi^2}[e^1\cos n\pi - e^{-1}\cos(-n\pi)] \\ &= \frac{(e^1-e^{-1})(-1)^n}{1+n^2\pi^2} = \frac{2(-1)^n\sinh(1)}{1+n^2\pi^2}, \end{aligned}$$

and,

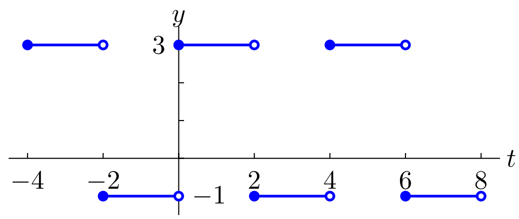
$$\begin{aligned} b_n &= \int_{-1}^{1} e^t \sin n\pi t\,dt \\ &= \frac{1}{1+n^2\pi^2}e^t[\sin n\pi t - n\pi \cos n\pi t]\Big|_{-1}^{1} \\ &= \frac{1}{1+n^2\pi^2}[e^1(-n\pi\cos n\pi) - e^{-1}(-n\pi\cos(-n\pi))] \\ &= \frac{(e^1-e^{-1})(-n\pi)(-1)^n}{1+n^2\pi^2} = \frac{2(-1)^n(-n\pi)\sinh(1)}{1+n^2\pi^2}. \end{aligned}$$

Therefore, the Fourier series is

$$f(t) \sim \sinh(1) + 2\sinh(1)\sum_{n=1}^{\infty} \frac{(-1)^n(\cos n\pi t - n\pi\sin n\pi t)}{1+n^2\pi^2}.$$

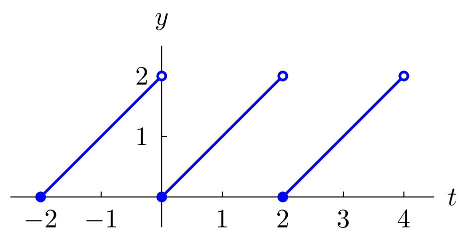
SECTION 10.3

1. (a)



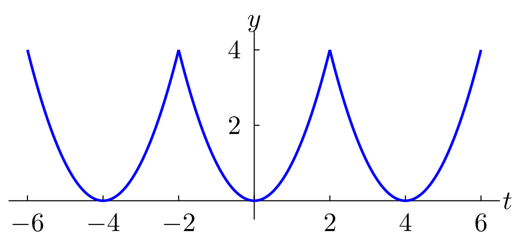
(b) All t except for $t = 2n$ for n an integer.
(c) For $t = 2n$, $f(t) = 3$ for n even and $f(t) = -1$ for n odd. Converges to $(3 + (-1))/2 = 1$ for all $t = 2n$.

3. (a)



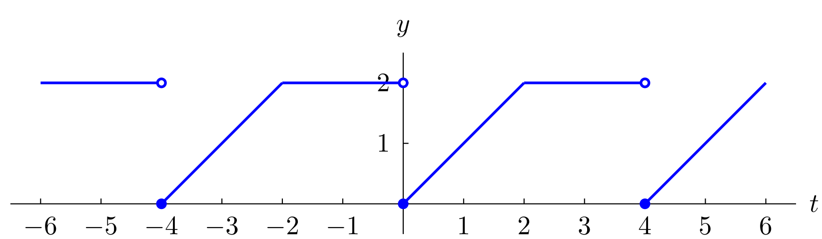
(b) All t except for $t = n$ for n an even integer.
(c) For t an even integer, $f(t) = 0$. Fourier series converges to 1.

5. (a)



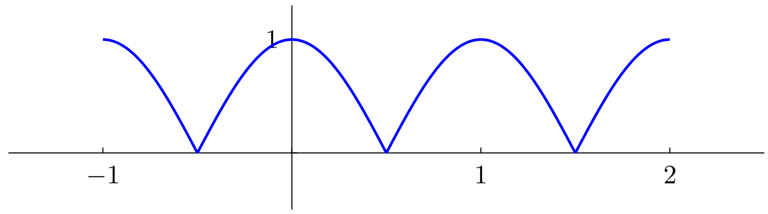
(b) All t since $f(t)$ is continuous for all t.
(c) No points of discontinuity.

7. (a)



(b) All t except for $t = 4n$ for n an integer.
(c) For t a multiple of 4, $f(t) = 0$. Fourier series converges to 1.

9. (a)



(b) All t since $f(t)$ is continuous.
(c) No points of discontinuity.

11. The Fourier series for the $2L$-periodic function $f(t) = t$ for $-L \le t < L$ is

$$f(t) \sim \frac{2L}{\pi}\left(\sin\frac{\pi}{L}t - \frac{1}{2}\sin\frac{2\pi}{L}t + \frac{1}{3}\sin\frac{3\pi}{L}t - \frac{1}{4}\sin\frac{4\pi}{L}t + \cdots\right)$$

This function is continuous for $-L < t < L$ so the Fourier series converges to $f(t)$ for $-L < t < L$. Letting $L = \pi$ gives an equality

$$t = 2\left(\sin t - \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t - \frac{1}{4}\sin 4t + \cdots\right), \qquad \text{for } -\pi < t < \pi.$$

Dividing by 2 gives the required identity. Substituting $t = \pi/2$ gives the summation.

13. The 2-periodic function defined by $f(t) = t^2$ for $-1 \le t \le 1$ has period 2 so $L = 1$. Compute the Fourier series of $f(t)$. The function is even, so the sine terms $b_n = 0$. For the cosine terms a_n:

$$a_0 = \int_{-1}^{1} f(t)\,dt = 2\int_0^1 f(t)\,dt = 2\int_0^1 t^2\,dt = 2\left.\frac{t^3}{3}\right|_0^1 = \frac{2}{3},$$

and for $n \ge 1$, (integration by parts is used multiple times)

$$\begin{aligned}
a_n &= \int_{-1}^{1} f(t)\cos n\pi t\,dt = 2\int_0^1 f(t)\cos n\pi t\,dt = 2\int_0^1 t^2\cos n\pi t\,dt \\
&= 2\left. t^2\cdot\frac{1}{n\pi}\sin n\pi t\right|_0^1 - 2\int_0^1 \frac{2t}{n\pi}\sin n\pi t\,dt = -\frac{4}{n\pi}\int_0^1 t\sin n\pi t\,dt \\
&= -\frac{4}{n\pi}\left[\left.\frac{-t}{n\pi}\cos n\pi t\right|_0^1 + \frac{1}{n\pi}\int_0^1 \cos n\pi t\,dt\right] \\
&= \frac{4}{n^2\pi^2}\cos n\pi - \left.\frac{4}{n^3\pi^3}\sin n\pi t\right|_0^1 \\
&= \frac{4}{n^2\pi^2}(-1)^n.
\end{aligned}$$

Therefore, the Fourier series is

$$f(t) \sim \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi t.$$

Since the function $f(t)$ is continuous for all t, the Fourier series converges to $f(t)$ for all t. In particular,

$$\frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi t = t^2, \qquad \text{for } -1 \leq t \leq 1.$$

15. $f(t)$ is 2π periodic and even. Thus the sine terms $b_n = 0$. For the cosine terms.

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} t^4 \, dt = \frac{2}{\pi} \int_0^{\pi} t^4 \, dt = \frac{2}{5}\pi^4.$$

For $n \geq 1$: The following integration formula, obtained by multiple integrations by parts, will be useful:

$$\int t^4 \cos at \, dt = \frac{1}{a} t^4 \sin at - \frac{1}{a^2} 4t^3 \cos at - \frac{1}{a^3} 12t^2 \sin at - \frac{1}{a^4} 24t \cos at + \frac{1}{a^5} 24 \sin at.$$

Then, since t^4 is even, and letting $a = n$ in the integration formula,

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} t^4 \cos nt \, dt = \frac{2}{\pi} \int_0^{\pi} t^4 \cos nt \, dt \\
&= \frac{2}{\pi} \left[\frac{1}{n} t^4 \sin nt + \frac{4}{n^2} t^3 \cos nt - \frac{12}{n^3} t^2 \sin nt - \frac{24}{n^4} t \cos nt + \frac{24}{n^5} \sin nt \right]_0^{\pi} \\
&= \frac{2}{\pi} \left[\frac{4}{n^2} \pi^3 \cos n\pi - \frac{24}{n^4} \pi \cos n\pi \right] \\
&= \frac{8}{n^2} \pi^2 (-1)^n - \frac{48}{n^4} (-1)^n.
\end{aligned}$$

Thus, the Fourier series is

$$f(t) \sim \frac{1}{5}\pi^4 + \sum_{n=1}^{\infty} \left[\frac{8}{n^2} \pi^2 (-1)^n - \frac{48}{n^4} (-1)^n \right] \cos nt.$$

Since $f(t)$ is continuous for all t, the Fourier series of $f(t)$ converges to $f(t)$ for all t. In particular, there is an identity

$$t^4 = \frac{1}{5}\pi^4 + \sum_{n=1}^{\infty} \left[\frac{8}{n^2} \pi^2 (-1)^n - \frac{48}{n^4} (-1)^n \right] \cos nt,$$

valid for all t. Setting $t=\pi$ gives

$$\pi^4 = \frac{1}{5}\pi^4 + \sum_{n=1}^{\infty} \frac{8}{n^2}\pi^2 - \sum_{n=1}^{\infty} \frac{48}{n^4}.$$

Thus,

$$\begin{aligned}
48\sum_{n=1}^{\infty} \frac{1}{n^4} &= -\frac{4}{5}\pi^4 + 8\pi^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \\
&= -\frac{4}{5}\pi^4 + 8\pi^2 \cdot \frac{\pi^2}{6} \qquad \text{from problem 13} \\
&= \pi^4\left(\frac{4}{3} - \frac{4}{5}\right) = \pi^4\left(\frac{8}{15}\right).
\end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \pi^4 \frac{8}{15\cdot 48} = \frac{\pi^4}{90}.$$

Setting $t=0$ gives

$$0 = \frac{1}{5}\pi^4 + \sum_{n=1}^{\infty} \frac{8}{n^2}\pi^2(-1)^n - \sum_{n=1}^{\infty} \frac{48}{n^4}(-1)^n.$$

Thus,

$$\begin{aligned}
48\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} &= \frac{\pi^4}{5} + 8\pi^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \\
&= \frac{\pi^4}{5} - 8\pi^2 \cdot \frac{\pi^2}{12} \qquad \text{from problem 13} \\
&= \frac{\pi^4}{5} - \frac{8\pi^4}{12} = \pi^4\left(\frac{1}{5} - \frac{2}{3}\right) = -\frac{7}{15}\pi^4.
\end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{15\cdot 48} = \frac{7\pi^4}{720}.$$

Section 10.4

1. Cosine series:

$$a_0 = \frac{2}{L}\int_0^L f(t)\,dt = \frac{2}{L}\int_0^L 1\,dt = 2,$$

and for $n \geq 1$

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(t)\cos\frac{n\pi t}{L}\,dt \\ &= \frac{2}{L}\int_0^L \cos\frac{n\pi t}{L}\,dt = \frac{2}{n\pi}\sin\frac{n\pi t}{L}\Big|_0^L = 0. \end{aligned}$$

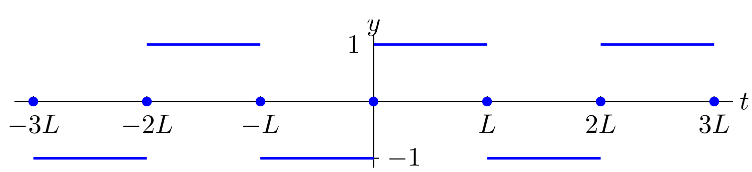
Thus, the Fourier cosine series is $f(t) \sim 1$ and this series converges to the constant function 1.
Sine series: For $n \geq 1$

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(t)\sin\frac{n\pi t}{L}\,dt \\ &= \frac{2}{L}\int_0^L \sin\frac{n\pi t}{L}\,dt = -\frac{2}{n\pi}\cos\frac{n\pi t}{L}\Big|_0^L \\ &= -\frac{2}{n\pi}(\cos n\pi - 1) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4}{n\pi} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Thus, the Fourier sine series is

$$f(t) \sim \frac{4}{\pi}\sum_{n\text{odd}} \frac{1}{n}\sin\frac{n\pi t}{L}.$$

This converges to the odd extension of $f(t)$, which is the odd square wave function (see Figure 10.5). The graph is



3. Cosine series: For $n = 0$,

$$\begin{aligned} a_0 &= \frac{2}{2}\int_0^2 f(t)\,dt \\ &= \int_0^2 t\,dt = \frac{t^2}{2}\Big|_0^2 = 2. \end{aligned}$$

For $n \geq 1$, taking advantage of the integration by parts formula

$$\int x\cos x\,dx = x\sin x + \cos x + C,$$

$$\begin{aligned} a_n &= \frac{2}{2}\int_0^2 f(t)\cos\frac{n\pi}{2}t\,dt \\ &= \int_0^2 t\cos\frac{n\pi}{2}t\,dt \qquad \left(\text{let } x = \frac{n\pi}{2}t \text{ so } t = \frac{2x}{n\pi} \text{ and } dt = \frac{2dx}{n\pi}\right) \\ &= \int_0^{n\pi} \frac{2x}{n\pi}\cos x\,\frac{2dx}{n\pi} = \frac{4}{n^2\pi^2}\left[x\sin x + \cos x\right]_{x=0}^{x=n\pi} \\ &= \frac{4}{n^2\pi^2}[\cos n\pi - 1] = \frac{2}{n^2\pi^2}[(-1)^n - 1] \end{aligned}$$

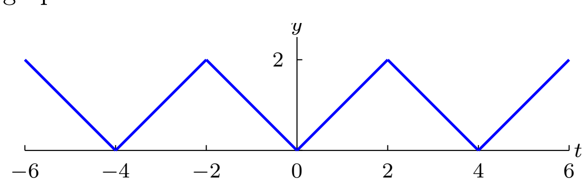
Therefore,

$$a_n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ -\frac{8}{n^2\pi^2} & \text{if } n \text{ is odd} \end{cases}$$

and the Fourier cosine series is

$$\begin{aligned} f(t) &\sim 1 - \frac{8}{\pi^2}\left(\frac{\cos\frac{\pi}{2}t}{1^2} + \frac{\cos\frac{3\pi}{2}t}{3^2} + \frac{\cos\frac{5\pi}{2}t}{5^2} + \frac{\cos\frac{7\pi}{2}t}{7^2} + \cdots\right) \\ &= 1 - \frac{8}{\pi^2}\sum_{n=\text{odd}} \frac{\cos\frac{n\pi}{2}t}{n^2}. \end{aligned}$$

This converges to the even extension of $f(t)$, which is an even triangular wave with graph



Sine series: For $n \geq 1$, taking advantage of the integration by parts formula

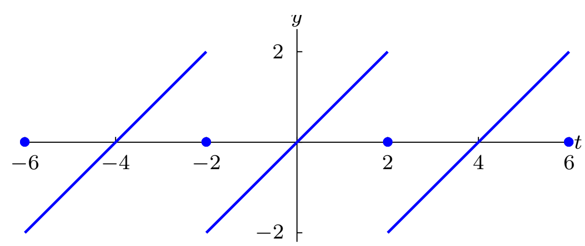
$$\int x\sin x\,dx = -x\cos x + \sin x + C,$$

$$\begin{aligned} b_n &= \frac{2}{2}\int_0^2 f(t)\sin\frac{n\pi}{2}t\,dt \\ &= \int_0^2 t\sin\frac{n\pi}{2}t\,dt \qquad \left(\text{let } x = \frac{n\pi}{2}t \text{ so } t = \frac{2x}{n\pi} \text{ and } dt = \frac{2dx}{n\pi}\right) \\ &= \int_0^{n\pi} \frac{2x}{n\pi}\sin x\,\frac{2dx}{n\pi} = \frac{4}{n^2\pi^2}\left[-x\cos x + \sin x\right]_{x=0}^{x=n\pi} \\ &= -\frac{4}{n\pi}\cos n\pi = -\frac{4}{n\pi}(-1)^n \end{aligned}$$

Therefore, the Fourier sine series is

$$\begin{aligned} f(t) &\sim \frac{4}{\pi}\left(\frac{\sin\frac{\pi}{2}t}{1} - \frac{\sin\frac{2\pi}{2}t}{2} + \frac{\sin\frac{3\pi}{2}t}{3} - \frac{\sin\frac{4\pi}{2}t}{4} + \cdots\right) \\ &= \frac{4}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin\frac{n\pi}{2}t}{n}. \end{aligned}$$

This converges to the odd extension of $f(t)$, which is a sawtooth wave with graph



5. Cosine series: For $n = 0$:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(t)\,dt = \frac{2}{\pi}\int_0^{\pi/2} dt = 1,$$

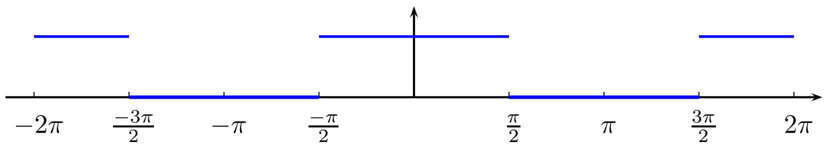
and for $n \geq 1$,

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} f(t)\cos nt\,dt = \frac{2}{\pi}\int_0^{\pi/2} cosnt\,dt \\ &= \frac{2}{n\pi}\sin nt\Big|_0^{\pi/2} = \frac{2}{n\pi}\sin\frac{n\pi}{2}. \end{aligned}$$

Thus, the Fourier cosine series is

$$f(t) \sim \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n}\cos nt = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{2k+1}\cos(2k+1)t.$$

This converges to the even extension of $f(t)$, which has the graph



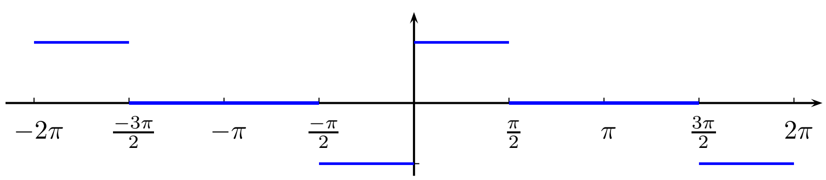
Sine series: For $n \geq 1$,

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(t) \sin nt \, dt = \frac{2}{\pi} \int_0^{\pi/2} sinnt \, dt$$
$$= \frac{-2}{n\pi} \cos nt \Big|_0^{\pi/2} = \frac{-2}{n\pi} (\cos \frac{n\pi}{2} - 1).$$

Thus, the Fourier sine series is

$$f(t) \sim \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{-2}{n\pi} (\cos \frac{n\pi}{2} - 1) \sin nt.$$

This converges to the odd extension of $f(t)$, which has the graph



7. Cosine series: For $n = 0$,

$$a_0 = \frac{2}{1} \int_0^1 f(t) \, dt = 2 \int_0^1 (t - t^2) \, dt = 2 \left[\frac{t^2}{2} - \frac{t^2}{3} \right]_0^1 = \frac{1}{3}.$$

For $n \geq 1$, taking advantage of the formula (obtained from repeated integration by parts):

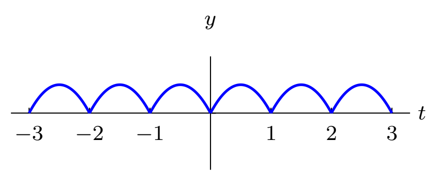
$$\int p(t) \cos at \, dt = \frac{1}{a} p(t) \sin at - \frac{1}{a} \int p'(t) \sin at \, dt$$
$$= \frac{1}{a} p(t) \sin at + \frac{1}{a^2} p'(t) \cos at - \frac{1}{a^3} p''(t) \sin at - \cdots$$
$$(+ + - - + + - - \cdots)(\text{signs alternate in pairs}),$$

$$\begin{aligned}
a_n &= 2\int_0^1 f(t)\cos n\pi t\,dt = 2\int_0^1 (t-t^2)\cos n\pi t\,dt \\
&= 2\left[\frac{1}{n\pi}(t-t^2)\sin n\pi t + \frac{1}{n^2\pi^2}(1-2t)\cos n\pi t - \frac{1}{n^3\pi^3}(-2)\sin n\pi t\right]_0^1 \\
&= 2\left[\frac{-1}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right] = \frac{-2}{n^2\pi^2}(\cos n\pi + 1) \\
&= \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{-4}{n^2\pi^2} & \text{if } n \text{ is even} \end{cases}
\end{aligned}$$

Therefore, the Fourier cosine series is

$$f(t) \sim \frac{1}{6} - \frac{4}{\pi^2}\sum_{n=\text{even}} \frac{\cos n\pi t}{n^2}.$$

This converges to the even extension of $f(t)$, which has the graph



Sine series: For $n \geq 1$, taking advantage of the formula (obtained from repeated integration by parts):

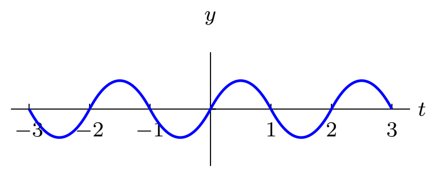
$$\begin{aligned}
\int p(t)\sin at\,dt &= -\frac{1}{a}p(t)\cos at + \frac{1}{a}\int p'(t)\cos at\,dt \\
&= -\frac{1}{a}p(t)\cos at + \frac{1}{a^2}p'(t)\sin at + \frac{1}{a^3}p''(t)\cos at - \cdots \\
&\quad (-++--++\cdots)(\text{signs alternate in pairs after first term}),
\end{aligned}$$

$$\begin{aligned}
b_n &= 2\int_0^1 f(t)\sin n\pi t\,dt = 2\int_0^1 (t-t^2)\sin n\pi t\,dt \\
&= 2\left[-\frac{1}{n\pi}(t-t^2)\cos n\pi t + \frac{1}{n^2\pi^2}(1-2t)\sin n\pi t + \frac{1}{n^3\pi^3}(-2)\cos n\pi t\right]_0^1 \\
&= \frac{-4}{n^3\pi^3}(\cos n\pi - 1) \\
&= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{8}{n^3\pi^3} & \text{if } n \text{ is odd} \end{cases}
\end{aligned}$$

Therefore, the Fourier sine series is

$$f(t) \sim \frac{8}{\pi^3} \sum_{n=\text{odd}} \frac{\sin n\pi t}{n^3}.$$

This converges to the odd extension of $f(t)$, which has the graph



9. Cosine series: The even extension of the function $f(t) = \cos t$ on $0 < t < \pi$ is just the cosine function on the whole real line. Thus, $f(t)$ it its own Fourier cosine series $f(t) \sim \cos t$, which converges to the cosine function.

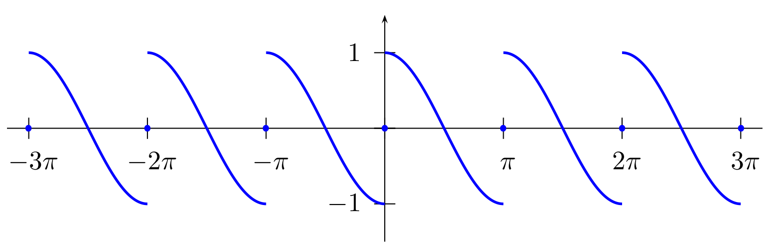
Sine series: For $n \geq 1$,

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^\pi f(t) \sin nt \, dt = \frac{2}{\pi} \int_0^\pi \cos t \sin nt \, dt \\ &= \frac{-2}{\pi} \left[\frac{1}{n^2 - 1} (\sin t \sin nt + n \cos t \cos nt) \right]_0^\pi \\ &= \frac{-2n}{\pi(n^2 - 1)} (\cos \pi \cos n\pi - 1) \\ &= \begin{cases} \frac{4n}{\pi(n^2-1)} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

Therefore, the Fourier sine series is

$$f(t) \sim \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=\text{even}} \frac{n}{n^2 - 1} \sin nt.$$

This converges to the odd extension of $f(t)$, which has the graph



11. Cosine series: For $n = 0$,

$$a_0 = \frac{2}{L}\int_0^L f(t)\,dt = \frac{2}{L}\int_0^L \big(1-\frac{2}{L}t\big)\,dt = \frac{2}{L}\big(t-\frac{t^2}{L}\big)\Big|_0^L = 0.$$

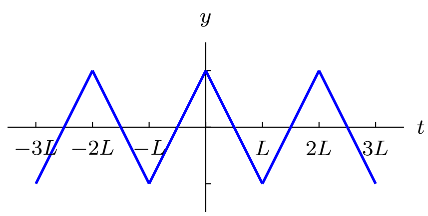
For $n \geq 1$,

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(t)\cos\frac{n\pi}{L}t\,dt = \frac{2}{L}\int_0^L \big(1-\frac{2}{L}t\big)\cos\frac{n\pi}{L}t\,dt \\ &= \frac{2}{L}\left[\frac{L}{n\pi}\big(1-\frac{2}{L}t\big)\sin\frac{n\pi}{L}t + \frac{L^2}{n^2\pi^2}\big(-\frac{2}{L}\big)\cos\frac{n\pi}{L}t\right]_0^L \\ &= -\frac{4}{n^2\pi^2}(\cos n\pi - 1) = \begin{cases} \frac{8}{n^2\pi^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \end{aligned}$$

Therefore, the Fourier cosine series is

$$f(t) \sim \frac{-4}{\pi^2}\sum_{n=\text{odd}} \frac{\cos\frac{n\pi}{L}t}{n^2}.$$

This converges to the even extension of $f(t)$, which has the graph



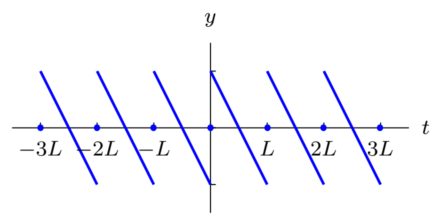
Sine series: For $n \geq 1$,

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(t)\sin\frac{n\pi}{L}t\,dt = \frac{2}{L}\int_0^L \big(1-\frac{2}{L}t\big)\sin\frac{n\pi}{L}t\,dt \\ &= \frac{2}{L}\left[\frac{-L}{n\pi}\big(1-\frac{2}{L}t\big)\cos\frac{n\pi}{L}t + \frac{L^2}{n^2\pi^2}\big(-\frac{2}{L}\big)\sin\frac{n\pi}{L}t\right]_0^L \\ &= \frac{2}{n\pi}\cos n\pi - \frac{-2}{n\pi} = \frac{2}{n\pi}((-1)^n + 1) \\ &= \begin{cases} \frac{4}{n\pi} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

Therefore, the Fourier sine series is

$$f(t) \sim \frac{4}{\pi}\sum_{n=\text{even}} \frac{\sin\frac{n\pi}{L}t}{n}.$$

This converges to the odd extension of $f(t)$, which has the graph



Section 10.5

1. The procedure is to write each of these functions as a linear combination of $f_1(t)$ and $f_2(t)$ (or other basic functions whose Fourier series are already computed) and then use Theorem 1.

(a) $f_3(t) = 1 - f_1(t)$. Thus,

$$f_3(t) = 1 - f_1(t) \sim \frac{1}{2} - \frac{2}{\pi} \sum_{n=\text{odd}} \frac{\sin nt}{n}.$$

(b) From Example 5 of Section 10.2, the Fourier series of the 2π-periodic sawtooth wave function $f(t) = t$ for $-\pi < t < \pi$, is

$$f(t) \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nt.$$

Since, $f_4(t) = f(t) - f_2(t)$,

$$\begin{aligned} f_4(t) &\sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nt - \left(\frac{\pi}{4} - \frac{2}{\pi} \sum_{n=\text{odd}} \frac{\cos nt}{n^2} + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nt}{n} \right) \\ &= -\frac{\pi}{4} + \frac{2}{\pi} \sum_{n=\text{odd}} \frac{\cos nt}{n^2} + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nt}{n}. \end{aligned}$$

(c) $f_5(t) = f_3(t) + f_2(t)$. Thus,

$$\begin{aligned} f_5(t) &\sim \frac{1}{2} - \frac{2}{\pi} \sum_{n=\text{odd}} \frac{\sin nt}{n} + \frac{\pi}{4} - \frac{2}{\pi} \sum_{n=\text{odd}} \frac{\cos nt}{n^2} + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nt}{n} \\ &= \frac{\pi}{4} + \frac{1}{2} - \frac{2}{\pi} \sum_{n=\text{odd}} \frac{\cos nt}{n^2} + \sum_{n=\text{odd}} \frac{-2+\pi}{\pi} \frac{\sin nt}{n} - \sum_{n=\text{even}} \frac{\sin nt}{n} \end{aligned}$$

(d) $f_6(t) = 2f_3(t)$. Thus,

$$f_6(t) \sim 2\left(\frac{1}{2} - \frac{2}{\pi}\sum_{n=\text{odd}} \frac{\sin nt}{n}\right) = 1 - \frac{4}{\pi}\sum_{n=\text{odd}} \frac{\sin nt}{n}$$

(e) $f_7(t) = 2f_3(t) + 3f_1(t) = 2(1 - f_1(t)) + 3f_1(t) = 2 + f_1(t)$. Thus,

$$f_7(t) = 2 + f_1(t) \sim \frac{5}{2} + \frac{2}{\pi}\sum_{n=\text{odd}} \frac{\sin nt}{n}.$$

(f) $f_8(t) = 1 + 2f_2(t)$. Thus,

$$f_8(t) \sim 1 + \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=\text{odd}} \frac{\cos nt}{n^2} + 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nt}{n}.$$

(g)

$$\begin{aligned} f_9(t) &= af_3(t) + bf_4(t) + cf_1(t) + df_2(t) \\ &= a(1 - f_1(t)) + b(t - f_2(t)) + cf_1(t) + df_2(t) \\ &= a + bt + (c - a)f_1(t) + (d - b)f_2(t). \end{aligned}$$

Thus,

$$\begin{aligned} f_9(t) &\sim a + b\left(2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nt\right) + c\left(\frac{1}{2} + \frac{2}{\pi}\sum_{n=\text{odd}}\frac{\sin nt}{n}\right) \\ &\quad + d\left(\frac{\pi}{4} - \frac{2}{\pi}\sum_{n=\text{odd}}\frac{\cos nt}{n^2} + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nt}{n}\right) \\ &= a + \frac{c}{2} + \frac{\pi d}{4} - \frac{2d}{\pi}\sum_{n=\text{odd}}\frac{\cos nt}{n^2} - \sum_{n=\text{even}}(2b + d)\frac{\sin nt}{n} \\ &\quad + \left(\frac{2c}{\pi} + 2b + d\right)\sum_{n=\text{odd}}\frac{\sin nt}{n}. \end{aligned}$$

3. The function $g(t) = |t| - \frac{\pi}{2}$ for $-\pi < t < \pi$ has the cosine term $a_0 = 0$ in its Fourier series, so the Fourier series of $\int_{-\pi}^{t} g(x)\,dx$ can be computed by termwise integration of the Fourier series of $g(t)$. For $-\pi < t \le 0$,

$$\begin{aligned} \int_{-\pi}^{t} g(x)\,dx &= \int_{-\pi}^{t}\left(|x| - \frac{\pi}{2}\right)dx = \int_{-\pi}^{t}\left(-x - \frac{\pi}{2}\right)dx \\ &= \left[-\frac{x^2}{2} - \frac{\pi}{2}x\right]_{-\pi}^{t} = -\frac{t^2}{2} - \frac{\pi}{2}t. \end{aligned}$$

For $0 < t < \pi$,

$$\int_{-\pi}^{t} g(x)\,dx = \int_{\pi}^{0} g(x)\,dx + \int_{0}^{t} g(x)\,dx = 0 + \int_{0}^{t} \left(|x| - \frac{\pi}{2}\right) dx$$
$$= \int_{0}^{t} \left(x - \frac{\pi}{2}\right) dx = \left[\frac{x^2}{2} - \frac{\pi}{2}x\right]_0^t = \frac{t^2}{2} - \frac{\pi}{2}t.$$

Thus,

$$\int_{-\pi}^{t} g(x)\,dx = \frac{1}{2}t^2 \operatorname{sgn} t - \frac{\pi}{2}t.$$

Theorem 7 applies to give

$$\int_{-\pi}^{t} g(x)\,dx \sim \frac{A_0}{2} - \frac{4}{\pi} \sum_{n=\text{odd}} \frac{1}{n^3} \sin nt.$$

Since $\int_{-\pi}^{t} g(x)\,dx$ is an odd function, the cosine term $A_0 = 0$. Solving for $f(t)$ gives

$$f(t) = 2\int_{-\pi}^{t} g(x)\,dx + \pi t.$$

Thus, using the known Fourier series for t given in Exercise 2, the Fourier series of $f(t)$ is given by

$$f(t) \sim -\frac{8}{\pi} \sum_{n=\text{odd}} \frac{1}{n^3} \sin nt + 2\pi \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nt$$
$$= \sum_{n=\text{odd}} \left(\frac{-8}{\pi n^3} + \frac{2\pi}{n}\right) \sin nt - \sum_{n=\text{even}} \frac{1}{n} \sin nt.$$

5. (a) $f(t)$ is continuous for $-2 < t < 0$ and for $0 < t < 2$ since it is defined by a polynomial on each of those open intervals. $\lim_{t\to 0^+} f(t) = \lim_{t\to 0^+} \frac{t^2}{2} - \frac{t}{2} = 0$ and $\lim_{t\to 0^-} f(t) = \lim_{t\to 0^-} -t/2 = 0$. Thus, $f(t)$ is continuous at 0. Since $\lim_{t\to 2^-} f(t) = \lim_{t\to 2^-} \frac{t^2}{2} - \frac{t}{2} = \frac{4}{2} - \frac{2}{2} = 1$ and $\lim_{t\to 2^+} f(t) = \lim_{t\to -2^+} f(t) = \lim_{t\to -2^+} -t/2 = 1$, it follows that $f(t)$ is continuous at 2, and similarly at -2. Since $f(t)$ is 4-periodic, it is thus continuous everywhere.
$f'(t) = \begin{cases} -\frac{1}{2} & \text{if } -2 < t < 0 \\ t - \frac{1}{2} & \text{if } 0 < t < 2 \end{cases}$ and $f''(t) = \begin{cases} 0 & \text{if } -2 < t < 0 \\ 1 & \text{if } 0 < t < 2 \end{cases}$ Thus, both $f'(t)$ and $f''(t)$ are piecewise continuous,and hence $f(t)$ is piecewise smooth. Therefore, the hypotheses of Theorem 3 are satisfied.

(b) Using Theorem 3 we can differentiate the Fourier series of $f(t)$ term by term to get

$$f'(t) \sim \frac{2}{\pi} \sum_{n=1}^{\infty} \left(-\frac{(-1)^n}{n} \sin \frac{n\pi}{2}t + \frac{(-1)^n - 1}{\pi n^2} \cos \frac{n\pi}{2}t\right).$$

(c) Since $\lim_{t\to 2^-} f'(t) = \lim_{t\to 2^-} t - \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2}$ and $\lim_{t\to 2^+} f'(t) = \lim_{t\to -2^+} f'(t) = \lim_{t\to -2^+} -\frac{1}{2} = -\frac{1}{2}$, it follows that $f'(t)$ is not continuous at 2, and similarly at -2. Thus, the hypotheses of Theorem 3 are not satisfied.

Section 10.6

1. If $g(t)$ is the 2-periodic square wave function defined on $-1 < t < 1$ by $g(t) = \begin{cases} -1 & \text{if } -1 < t < 0 \\ 1 & \text{if } 0 < t < 1 \end{cases}$ then $f(t) = \frac{1}{2} + \frac{1}{2}g(t)$. Thus, the Fourier series of $f(t)$ is

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=\text{odd}} \frac{\sin n\pi t}{n}.$$

Let $y(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n \cos n\pi t + B_n \sin n\pi t)$ be a 2-periodic solution of $y'' + 4y = f(t)$ expressed as the sum of its Fourier series. Then $y(t)$ will satisfy the hypotheses of Theorem 3 of Section 10.5. Thus, differentiating twice will give

$$y''(t) = \sum_{n=1}^{\infty}(-n^2\pi^2 A_n \cos n\pi t - n^2\pi^2 B_n \sin n\pi t).$$

Substituting into the differential equation gives

$$\begin{aligned} y''(t) + 4y(t) &= 2A_0 + \sum_{n=1}^{\infty}(A_n(4 - n^2\pi^2)\cos n\pi t + B_n(4 - n^2\pi^2)\sin n\pi t) \\ &= \frac{1}{2} + \frac{2}{\pi}\sum_{n=\text{odd}} \frac{\sin n\pi t}{n}. \end{aligned}$$

Comparing corresponding coefficients of $\cos n\pi t$ and $\sin n\pi t$ gives the equations

$$\begin{aligned} 2A_0 &= \frac{1}{2} \\ A_n(4 - n^2\pi^2) &= 0 \qquad \text{for all } n \geq 1 \\ B_n(4 - n^2\pi^2) &= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{2}{n\pi} & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

Solving these equations gives $A_0 = 1/4$, $A_n = 0$ for all n, $B_n = 0$ for n even, and for n odd,

$$B_n = \frac{2}{(4 - n^2\pi^2)n\pi}.$$

Thus, the unique 2-periodic solution is the sum of the Fourier series expansion

$$y(t) = \frac{1}{8} + \frac{2}{\pi} \sum_{n=\text{odd}} \frac{1}{n(4 - n^2\pi^2)} \sin n\pi t.$$

3. The characteristic polynomial $q(s) = s^2 + 1$ has a root $i = in\omega$ for $n = 1$, so Theorem 2 does not apply. However, writing $\sum_{n=1}^{\infty} n^{-2} \cos nt = \cos t + \sum_{n=2}^{\infty} n^{-2} \cos nt$ and solving the two equations $y'' + y = \cos t$ and $y'' + y = f(t)$ separately, the original equation can be solved by linearity. Start with $y''+y = \cos t$. This can be solved by undetermined coefficients. Since $q(s) = n^2 + 1$ and $\mathcal{L}\{\cos t\} = s^2 + 1$, a test function has the form $y(t) = At \cos t + Bt \sin t$. Then $y'(t) = A \cos t - At \sin t + B \sin t + Bt \cos t$, and $y''(t) = -2A \sin t - At \cos t + 2B \cos t - Bt \sin t$. Substituting into $y'' + y = \cos t$ gives

$$-2A \sin t + 2B \cos t = \cos t.$$

Equating coefficients of $\sin t$ and $\cos t$ gives $A = 0$ and $B = 1/2$. Thus, a particular solution of $y'' + y = \cos t$ is $y_1(t) = \frac{1}{2} t \sin t$. Now find a particular solution of $y'' + y = f(t)$ by looking for a periodic solution $y_2(t) = \sum_{n=2}^{\infty} (A_n \cos nt + B_n \sin nt)$. Substitute into the differential equation to get

$$y_2'' + y_2 = \sum_{n=2}^{\infty} (A_n(1 - n^2) \cos nt + B_n(1 - n^2) \sin nt) = \sum_{n=2}^{\infty} \frac{1}{n^2} \cos nt.$$

Comparing coefficients of $\cos nt$ and $\sin nt$ gives $B_n = 0$ and $A_n = \frac{1}{n^2(1-n^2)}$, so that a particular solution of $y'' + y = f(t)$ is

$$y_2(t) = \sum_{n=2}^{\infty} \frac{1}{n^2(1 - n^2)} \cos nt.$$

By linearity, a particular solution of the original equation is

$$y_p(t) = \frac{1}{2} t \sin t + \sum_{n=2}^{\infty} \frac{1}{n^2(1 - n^2)} \cos nt,$$

and the general solution is

$$y_g(t) = y_h(t) + y_p(t) = C_1 \cos t + C_2 \sin t + \frac{1}{2} t \sin t + \sum_{n=2}^{\infty} \frac{1}{n^2(1 - n^2)} \cos nt.$$

5. $f(t)$ is the even extension of the function defined on the interval (0, 2) by $f(t) = 5$ if $0 < t < 1$ and $f(t) = 0$ if $1 < t < 2$. Thus the Fourier series is a cosine series with

$$a_0 = \frac{2}{2}\int_0^2 f(t)\,dt = \int_0^1 5\,dt = 5,$$

and for $n \geq 1$

$$\begin{aligned} a_n &= \frac{2}{2}\int_0^2 f(t)\cos\frac{n\pi t}{2}\,dt = \int_0^1 \cos\frac{n\pi t}{2}\,dt \\ &= \frac{2}{n\pi}\sin\frac{n\pi t}{2}\bigg|_0^1 = \frac{2}{n\pi}\sin\frac{n\pi}{2}. \end{aligned}$$

Hence,

$$a_n = \begin{cases} 0 & \text{if } n = 2k \text{ for } k \geq 1, \\ (-1)^k & \text{if } n = 2k+1 \text{ for } k \geq 0. \end{cases}$$

Thus, the Fourier series of the forcing function is

$$f(t) \sim \frac{5}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^k \frac{\cos\frac{(2k+1)\pi t}{2}}{2k+1}.$$

Let $y(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n \cos\frac{n\pi t}{2} + B_n \sin\frac{n\pi t}{2})$ be a 4-periodic solution of $y'' + 10y = f(t)$ expressed as the sum of its Fourier series. Then $y(t)$ will satisfy the hypotheses of Theorem 3 of Section 10.5. Thus, differentiating twice will give

$$y''(t) = \sum_{n=1}^{\infty}\left[-\frac{n^2\pi^2}{4}A_n\cos\frac{n\pi t}{2} - \frac{n^2\pi^2}{4}B_n\sin\frac{n\pi t}{2}\right].$$

Substituting into the differential equation gives

$$\begin{aligned} y''(t) + 10y(t) &= 5A_0 + \sum_{n=1}^{\infty}\left[A_n(10 - \frac{n^2\pi^2}{4})\cos\frac{n\pi t}{2} + B_n(10 - \frac{n^2\pi^2}{4})\sin\frac{n\pi t}{2}\right] \\ &= \frac{5}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^k\frac{\cos\frac{(2k+1)\pi t}{2}}{2k+1}. \end{aligned}$$

Comparing corresponding coefficients of $\cos n\pi t$ and $\sin n\pi t$ gives the equations

$$\begin{aligned} 5A_0 &= \frac{5}{2} \\ A_n\big(10 - \frac{n^2\pi^2}{4}\big) &= 0 \qquad \text{for all even } n \geq 1 \\ A_{2k-1}\big(10 - \frac{(2k+1)^2\pi^2}{4}\big) &= \frac{2(-1)^k}{\pi(2k+1)} \qquad \text{for } k \geq 1 \\ B_n\big(10 - \frac{n^2\pi^2}{4}\big) &= 0 \qquad \text{for all } n \geq 1 \end{aligned}$$

Solving these equations gives $A_0 = 1/2$, $B_n = 0$ for all n, $A_n = 0$ for n even, and for $n = 2k+1$ odd,

$$A_n = A_{2k+1} = \frac{2(-1)^k}{(10 - \frac{n^2\pi^2}{4})(2k+1)\pi}.$$

Thus, the unique 4-periodic solution is the sum of the Fourier series expansion

$$y(t) = \frac{1}{4} + \sum_{k=1}^{\infty} \frac{2(-1)^k}{(10 - \frac{n^2\pi^2}{4})(2k+1)\pi} \cos\frac{n\pi t}{2}.$$

7. The Fourier series of $f(t)$ is the cosine series of $f(t)$. It was computed in Exercise 2 of Section 10.4 as $f(t) \sim \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=\text{odd}} \frac{\cos n\pi t}{n^2}$. Let $y(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n \cos n\pi t + B_n \sin n\pi t)$ be a 2-periodic solution of $y'' + 5y = f(t)$ expressed as the sum of its Fourier series. Then $y(t)$ will satisfy the hypotheses of Theorem 3 of Section 10.5. Thus, differentiating twice will give

$$y''(t) = \sum_{n=1}^{\infty} \left[-n^2\pi^2 A_n \cos n\pi t - n^2\pi^2 B_n \sin n\pi t\right].$$

Substituting into the differential equation gives

$$\begin{aligned} y''(t) + y(t) &= \frac{5A_0}{2} + \sum_{n=1}^{\infty} \left[A_n(5 - n^2\pi^2)\cos n\pi t + B_n(5 - n^2\pi^2)\sin n\pi t\right] \\ &= \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=\text{odd}} \frac{\cos n\pi t}{n^2}. \end{aligned}$$

Comparing corresponding coefficients of $\cos n\pi t$ and $\sin n\pi t$ gives the equations

$$\begin{aligned}\frac{5A_0}{2} &= \frac{1}{2}\\ A_n(5-n^2\pi^2) &= 0 \qquad \text{for all even } n \geq 1\\ A_n(5-n^2\pi^2) &= \frac{4}{\pi^2 n^2} \qquad \text{for odd } n \geq 1\\ B_n(5-n^2\pi^2) &= 0 \qquad \text{for all } n \geq 1\end{aligned}$$

Solving these equations gives $A_0 = 1/5$, $B_n = 0$ for all n, $A_n = 0$ for n even, and for n odd,

$$A_n = \frac{4}{(5-n^2\pi^2)\pi^2 n^2}.$$

Thus, the unique 4-periodic solution is the sum of the Fourier series expansion

$$y(t) = \frac{1}{10} + \sum_{n=\text{odd}} \frac{4}{(5-n^2\pi^2)\pi^2 n^2} \cos n\pi t.$$