11.1 EIGENVALUE PROBLEMS FOR $y'' + \lambda y = 0$

In Chapter 12 we'll study partial differential equations that arise in problems of heat conduction, wave propagation, and potential theory. The purpose of this chapter is to develop tools required to solve these equations. In this section we consider the following problems, where λ is a real number and $L > 0$:

Problem 1: $\quad y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0$

Problem 2: $\quad y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(L) = 0$

Problem 3: $\quad y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(L) = 0$

Problem 4: $\quad y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(L) = 0$

Problem 5: $\quad y'' + \lambda y = 0, \quad y(-L) = y(L), \quad y'(-L) = y'(L)$

In each problem the conditions following the differential equation are called *boundary conditions*. Note that the boundary conditions in Problem 5, unlike those in Problems 1-4, don't require that y or y' be zero at the boundary points, but only that y have the same value at $x = \pm L$, and that y' have the same value at $x = \pm L$. We say that the boundary conditions in Problem 5 are *periodic*.

Obviously, $y \equiv 0$ (the trivial solution) is a solution of Problems 1-5 for any value of λ. For most values of λ, there are no other solutions. The interesting question is this:

For what values of λ does the problem have nontrivial solutions, and what are they?

A value of λ for which the problem has a nontrivial solution is an *eigenvalue* of the problem, and the nontrivial solutions are λ-*eigenfunctions*, or *eigenfunctions associated with λ*. Note that a nonzero constant multiple of a λ-eigenfunction is again a λ-eigenfunction.

Problems 1-5 are called *eigenvalue problems*. *Solving* an eigenvalue problem means finding all its eigenvalues and associated eigenfunctions. We'll take it as given here that all the eigenvalues of Problems 1-5 are real numbers. This is proved in a more general setting in Section 13.2.

Theorem 11.1.1 *Problems 1–5 have no negative eigenvalues. Moreover, $\lambda = 0$ is an eigenvalue of Problems 2 and 5, with associated eigenfunction $y_0 = 1$, but $\lambda = 0$ isn't an eigenvalue of Problems 1, 3, or 4.*

Proof We consider Problems 1-4, and leave Problem 5 to you (Exercise 1). If $y'' + \lambda y = 0$, then $y(y'' + \lambda y) = 0$, so

$$\int_0^L y(x)(y''(x) + \lambda y(x))\,dx = 0;$$

therefore,

$$\lambda \int_0^L y^2(x)\,dx = -\int_0^L y(x)y''(x)\,dx. \tag{11.1.1}$$

Integration by parts yields

$$\begin{aligned} \int_0^L y(x)y''(x)\,dx &= y(x)y'(x)\Big|_0^L - \int_0^L (y'(x))^2\,dx \\ &= y(L)y'(L) - y(0)y'(0) - \int_0^L (y'(x))^2\,dx. \end{aligned} \tag{11.1.2}$$

However, if y satisfies any of the boundary conditions of Problems 1-4, then

$$y(L)y'(L) - y(0)y'(0) = 0;$$

hence, (11.1.1) and (11.1.2) imply that

$$\lambda \int_0^L y^2(x)\,dx = \int_0^L (y'(x))^2\,dx.$$

If $y \not\equiv 0$, then $\int_0^L y^2(x)\,dx > 0$. Therefore $\lambda \ge 0$ and, if $\lambda = 0$, then $y'(x) = 0$ for all x in $(0, L)$ (why?), and y is constant on $(0, L)$. Any constant function satisfies the boundary conditions of Problem 2, so $\lambda = 0$ is an eigenvalue of Problem 2 and any nonzero constant function is an associated eigenfunction. However, the only constant function that satisfies the boundary conditions of Problems 1, 3, or 4 is $y \equiv 0$. Therefore $\lambda = 0$ isn't an eigenvalue of any of these problems.

Example 11.1.1 (Problem 1) Solve the eigenvalue problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0. \tag{11.1.3}$$

Solution From Theorem 11.1.1, any eigenvalues of (11.1.3) must be positive. If y satisfies (11.1.3) with $\lambda > 0$, then

$$y = c_1 \cos\sqrt{\lambda}\,x + c_2 \sin\sqrt{\lambda}\,x,$$

where c_1 and c_2 are constants. The boundary condition $y(0) = 0$ implies that $c_1 = 0$. Therefore $y = c_2 \sin\sqrt{\lambda}\,x$. Now the boundary condition $y(L) = 0$ implies that $c_2 \sin\sqrt{\lambda}\,L = 0$. To make $c_2 \sin\sqrt{\lambda}\,L = 0$ with $c_2 \ne 0$, we must choose $\sqrt{\lambda} = n\pi/L$, where n is a positive integer. Therefore $\lambda_n = n^2\pi^2/L^2$ is an eigenvalue and

$$y_n = \sin\frac{n\pi x}{L}$$

is an associated eigenfunction. ■

For future reference, we state the result of Example 11.1.1 as a theorem.

Theorem 11.1.2 *The eigenvalue problem*

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0$$

has infinitely many positive eigenvalues $\lambda_n = n^2\pi^2/L^2$*, with associated eigenfunctions*

$$y_n = \sin\frac{n\pi x}{L}, \quad n = 1, 2, 3, \dots.$$

There are no other eigenvalues.

We leave it to you to prove the next theorem about Problem 2 by an argument like that of Example 11.1.1 (Exercise 17).

Theorem 11.1.3 *The eigenvalue problem*

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(L) = 0$$

has the eigenvalue $\lambda_0 = 0$*, with associated eigenfunction* $y_0 = 1$*, and infinitely many positive eigenvalues* $\lambda_n = n^2\pi^2/L^2$*, with associated eigenfunctions*

$$y_n = \cos\frac{n\pi x}{L}, n = 1, 2, 3 \dots.$$

There are no other eigenvalues.

Example 11.1.2 (Problem 3) Solve the eigenvalue problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(L) = 0. \tag{11.1.4}$$

Solution From Theorem 11.1.1, any eigenvalues of (11.1.4) must be positive. If y satisfies (11.1.4) with $\lambda > 0$, then

$$y = c_1 \cos \sqrt{\lambda}\, x + c_2 \sin \sqrt{\lambda}\, x,$$

where c_1 and c_2 are constants. The boundary condition $y(0) = 0$ implies that $c_1 = 0$. Therefore $y = c_2 \sin \sqrt{\lambda}\, x$. Hence, $y' = c_2\sqrt{\lambda} \cos \sqrt{\lambda}\, x$ and the boundary condition $y'(L) = 0$ implies that $c_2 \cos \sqrt{\lambda}\, L = 0$. To make $c_2 \cos \sqrt{\lambda}\, L = 0$ with $c_2 \neq 0$ we must choose

$$\sqrt{\lambda} = \frac{(2n-1)\pi}{2L},$$

where n is a positive integer. Then $\lambda_n = (2n-1)^2\pi^2/4L^2$ is an eigenvalue and

$$y_n = \sin \frac{(2n-1)\pi x}{2L}$$

is an associated eigenfunction. ■

For future reference, we state the result of Example 11.1.2 as a theorem.

Theorem 11.1.4 *The eigenvalue problem*

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(L) = 0$$

has infinitely many positive eigenvalues $\lambda_n = (2n-1)^2\pi^2/4L^2$*, with associated eigenfunctions*

$$y_n = \sin \frac{(2n-1)\pi x}{2L}, \quad n = 1, 2, 3, \dots.$$

There are no other eigenvalues.

We leave it to you to prove the next theorem about Problem 4 by an argument like that of Example 11.1.2 (Exercise 18).

Theorem 11.1.5 *The eigenvalue problem*

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(L) = 0$$

has infinitely many positive eigenvalues $\lambda_n = (2n-1)^2\pi^2/4L^2$*, with associated eigenfunctions*

$$y_n = \cos \frac{(2n-1)\pi x}{2L}, \quad n = 1, 2, 3, \dots.$$

There are no other eigenvalues.

Example 11.1.3 (Problem 5) Solve the eigenvalue problem

$$y'' + \lambda y = 0, \quad y(-L) = y(L), \quad y'(-L) = y'(L). \tag{11.1.5}$$

Solution From Theorem 11.1.1, $\lambda = 0$ is an eigenvalue of (11.1.5) with associated eigenfunction $y_0 = 1$, and any other eigenvalues must be positive. If y satisfies (11.1.5) with $\lambda > 0$, then

$$y = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x, \tag{11.1.6}$$

where c_1 and c_2 are constants. The boundary condition $y(-L) = y(L)$ implies that

$$c_1 \cos(-\sqrt{\lambda}\, L) + c_2 \sin(-\sqrt{\lambda}\, L) = c_1 \cos\sqrt{\lambda}\, L + c_2 \sin\sqrt{\lambda}\, L. \tag{11.1.7}$$

Since

$$\cos(-\sqrt{\lambda}\, L) = \cos\sqrt{\lambda}\, L \quad \text{and} \quad \sin(-\sqrt{\lambda}\, L) = -\sin\sqrt{\lambda}\, L, \tag{11.1.8}$$

(11.1.7) implies that

$$c_2 \sin\sqrt{\lambda}\, L = 0. \tag{11.1.9}$$

Differentiating (11.1.6) yields

$$y' = \sqrt{\lambda}\left(-c_1 \sin\sqrt{\lambda} x + c_2 \cos\sqrt{\lambda} x\right).$$

The boundary condition $y'(-L) = y'(L)$ implies that

$$-c_1 \sin(-\sqrt{\lambda}\, L) + c_2 \cos(-\sqrt{\lambda}\, L) = -c_1 \sin\sqrt{\lambda}\, L + c_2 \cos\sqrt{\lambda}\, L,$$

and (11.1.8) implies that

$$c_1 \sin\sqrt{\lambda}\, L = 0. \tag{11.1.10}$$

Eqns. (11.1.9) and (11.1.10) imply that $c_1 = c_2 = 0$ unless $\sqrt{\lambda} = n\pi/L$, where n is a positive integer. In this case (11.1.9) and (11.1.10) both hold for arbitrary c_1 and c_2. The eigenvalue determined in this way is $\lambda_n = n^2\pi^2/L^2$, and each such eigenvalue has the linearly independent associated eigenfunctions

$$\cos\frac{n\pi x}{L} \quad \text{and} \quad \sin\frac{n\pi x}{L}. \qquad \blacksquare$$

For future reference we state the result of Example 11.1.3 as a theorem.

Theorem 11.1.6 *The eigenvalue problem*

$$y'' + \lambda y = 0, \quad y(-L) = y(L), \quad y'(-L) = y'(L),$$

has the eigenvalue $\lambda_0 = 0$, with associated eigenfunction $y_0 = 1$ and infinitely many positive eigenvalues $\lambda_n = n^2\pi^2/L^2$, with associated eigenfunctions

$$y_{1n} = \cos\frac{n\pi x}{L} \quad \textit{and} \quad y_{2n} = \sin\frac{n\pi x}{L}, \quad n = 1, 2, 3, \dots.$$

There are no other eigenvalues.

Orthogonality

We say that two integrable functions f and g are *orthogonal* on an interval $[a, b]$ if

$$\int_a^b f(x)g(x)\, dx = 0.$$

More generally, we say that the functions $\phi_1, \phi_2, \dots, \phi_n, \dots$ (finitely or infinitely many) are orthogonal on $[a, b]$ if

$$\int_a^b \phi_i(x)\phi_j(x)\, dx = 0 \quad \text{whenever} \quad i \neq j.$$

The importance of orthogonality will become clear when we study Fourier series in the next two sections.

Example 11.1.4 Show that the eigenfunctions

$$1,\ \cos\frac{\pi x}{L},\ \sin\frac{\pi x}{L},\ \cos\frac{2\pi x}{L},\ \sin\frac{2\pi x}{L},\dots,\cos\frac{n\pi x}{L},\ \sin\frac{n\pi x}{L},\dots \tag{11.1.11}$$

of Problem 5 are orthogonal on $[-L, L]$.

Solution We must show that

$$\int_{-L}^{L} f(x)g(x)\,dx = 0 \tag{11.1.12}$$

whenever f and g are distinct functions from (11.1.11). If r is any nonzero integer, then

$$\int_{-L}^{L} \cos\frac{r\pi x}{L}\,dx = \frac{L}{r\pi}\sin\frac{r\pi x}{L}\Big|_{-L}^{L} = 0. \tag{11.1.13}$$

and

$$\int_{-L}^{L} \sin\frac{r\pi x}{L}\,dx = -\frac{L}{r\pi}\cos\frac{r\pi x}{L}\Big|_{-L}^{L} = 0.$$

Therefore (11.1.12) holds if $f \equiv 1$ and g is any other function in (11.1.11).

If $f(x) = \cos m\pi x/L$ and $g(x) = \cos n\pi x/L$ where m and n are distinct positive integers, then

$$\int_{-L}^{L} f(x)g(x)\,dx = \int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx. \tag{11.1.14}$$

To evaluate this integral, we use the identity

$$\cos A\cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$$

with $A = m\pi x/L$ and $B = n\pi x/L$. Then (11.1.14) becomes

$$\int_{-L}^{L} f(x)g(x)\,dx = \frac{1}{2}\left[\int_{-L}^{L}\cos\frac{(m-n)\pi x}{L}\,dx + \int_{-L}^{L}\cos\frac{(m+n)\pi x}{L}\,dx\right].$$

Since $m-n$ and $m+n$ are both nonzero integers, (11.1.13) implies that the integrals on the right are both zero. Therefore (11.1.12) is true in this case.

If $f(x) = \sin m\pi x/L$ and $g(x) = \sin n\pi x/L$ where m and n are distinct positive integers, then

$$\int_{-L}^{L} f(x)g(x)\,dx = \int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx. \tag{11.1.15}$$

To evaluate this integral, we use the identity

$$\sin A\sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

with $A = m\pi x/L$ and $B = n\pi x/L$. Then (11.1.15) becomes

$$\int_{-L}^{L} f(x)g(x)\,dx = \frac{1}{2}\left[\int_{-L}^{L}\cos\frac{(m-n)\pi x}{L}\,dx - \int_{-L}^{L}\cos\frac{(m+n)\pi x}{L}\,dx\right] = 0.$$

If $f(x) = \sin m\pi x/L$ and $g(x) = \cos n\pi x/L$ where m and n are positive integers (not necessarily distinct), then

$$\int_{-L}^{L} f(x)g(x)\,dx = \int_{-L}^{L} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = 0$$

because the integrand is an odd function and the limits are symmetric about $x = 0$. ■

Exercises 19-22 ask you to verify that the eigenfunctions of Problems 1-4 are orthogonal on $[0, L]$. However, this also follows from a general theorem that we'll prove in Chapter 13.

11.1 Exercises

1. Prove that $\lambda = 0$ is an eigenvalue of Problem 5 with associated eigenfunction $y_0 = 1$, and that any other eigenvalues must be positive. HINT: *See the proof of Theorem* 11.1.1.

In Exercises 2-16 solve the eigenvalue problem.

2. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y(\pi) = 0$

3. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(\pi) = 0$

4. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(\pi) = 0$

5. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(\pi) = 0$

6. $y'' + \lambda y = 0, \quad y(-\pi) = y(\pi), \quad y'(-\pi) = y'(\pi)$

7. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(1) = 0$

8. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(1) = 0$

9. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) = 0$

10. $y'' + \lambda y = 0, \quad y(-1) = y(1), \quad y'(-1) = y'(1)$

11. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(1) = 0$

12. $y'' + \lambda y = 0, \quad y(-2) = y(2), \quad y'(-2) = y'(2)$

13. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y(2) = 0$

14. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(3) = 0$

15. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(1/2) = 0$

16. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(5) = 0$

17. Prove Theorem 11.1.3.

18. Prove Theorem 11.1.5.

19. Verify that the eigenfunctions

$$\sin\frac{\pi x}{L},\ \sin\frac{2\pi x}{L}, \dots,\ \sin\frac{n\pi x}{L}, \dots$$

of Problem 1 are orthogonal on $[0, L]$.

20. Verify that the eigenfunctions

$$1,\ \cos\frac{\pi x}{L},\ \cos\frac{2\pi x}{L}, \dots,\ \cos\frac{n\pi x}{L}, \dots$$

of Problem 2 are orthogonal on $[0, L]$.

CHAPTER 12
Fourier Solutions of Partial Differential

IN THIS CHAPTER we use the series discussed in Chapter 11 to solve partial differential equations that arise in problems of mathematical physics.

SECTION 12.1 deals with the partial differential equation

$$u_t = a^2 u_{xx},$$

which arises in problems of conduction of heat.

SECTION 12.2 deals with the partial differential equation

$$u_{tt} = a^2 u_{xx},$$

which arises in the problem of the vibrating string.

SECTION 12.3 deals with the partial differential equation

$$u_{xx} + u_{yy} = 0,$$

which arises in steady state problems of heat conduction and potential theory.

SECTION 12.4 deals with the partial differential equation

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0,$$

which is the equivalent to the equation studied in Section 1.3 when the independent variables are polar coordinates.

12.1 THE HEAT EQUATION

We begin the study of partial differential equations with the problem of heat flow in a uniform bar of length L, situated on the x axis with one end at the origin and the other at $x = L$ (Figure 12.1.1).

We assume that the bar is perfectly insulated except possibly at its endpoints, and that the temperature is constant on each cross section and therefore depends only on x and t. We also assume that the thermal properties of the bar are independent of x and t. In this case, it can be shown that the temperature $u = u(x,t)$ at time t at a point x units from the origin satisfies the partial differential equation

$$u_t = a^2 u_{xx}, \quad 0 < x < L, \quad t > 0,$$

where a is a positive constant determined by the thermal properties. This is the *heat equation*.

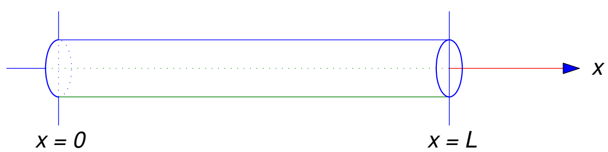

Figure 12.1.1 A uniform bar of length L

To determine u, we must specify the temperature at every point in the bar when $t = 0$, say

$$u(x,0) = f(x), \quad 0 \le x \le L.$$

We call this the *initial* condition. We must also specify *boundary conditions* that u must satisfy at the ends of the bar for all $t > 0$. We'll call this problem an *initial-boundary value problem.*

We begin with the boundary conditions $u(0,t) = u(L,t) = 0$, and write the initial-boundary value problem as

$$\begin{aligned} u_t &= a^2 u_{xx}, \quad 0 < x < L, \quad t > 0, \\ u(0,t) &= 0, \quad u(L,t) = 0, \quad t > 0, \\ u(x,0) &= f(x), \quad 0 \le x \le L. \end{aligned} \tag{12.1.1}$$

Our method of solving this problem is called *separation of variables* (not to be confused with method of separation of variables used in Section 2.2 for solving ordinary differential equations). We begin by looking for functions of the form

$$v(x,t) = X(x)T(t)$$

that are not identically zero and satisfy

$$v_t = a^2 v_{xx}, \quad v(0,t) = 0, \quad v(L,t) = 0$$

for all (x,t). Since

$$v_t = XT' \quad \text{and} \quad v_{xx} = X''T,$$

$v_t = a^2 v_{xx}$ if and only if

$$XT' = a^2 X''T,$$

which we rewrite as

$$\frac{T'}{a^2T} = \frac{X''}{X}.$$

Since the expression on the left is independent of x while the one on the right is independent of t, this equation can hold for all (x,t) only if the two sides equal the same constant, which we call a *separation constant*, and write it as $-\lambda$; thus,

$$\frac{X''}{X} = \frac{T'}{a^2T} = -\lambda.$$

This is equivalent to

$$X'' + \lambda X = 0$$

and

$$T' = -a^2\lambda T. \tag{12.1.2}$$

Since $v(0,t) = X(0)T(t) = 0$ and $v(L,t) = X(L)T(t) = 0$ and we don't want T to be identically zero, $X(0) = 0$ and $X(L) = 0$. Therefore λ must be an eigenvalue of the boundary value problem

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X(L) = 0, \tag{12.1.3}$$

and X must be a λ-eigenfunction. From Theorem 11.1.2, the eigenvalues of (12.1.3) are $\lambda_n = n^2\pi^2/L^2$, with associated eigenfunctions

$$X_n = \sin\frac{n\pi x}{L}, \quad n = 1, 2, 3, \dots.$$

Substituting $\lambda = n^2\pi^2/L^2$ into (12.1.2) yields

$$T' = -(n^2\pi^2a^2/L^2)T,$$

which has the solution

$$T_n = e^{-n^2\pi^2a^2t/L^2}.$$

Now let

$$v_n(x,t) = X_n(x)T_n(t) = e^{-n^2\pi^2a^2t/L^2}\sin\frac{n\pi x}{L}, \quad n = 1, 2, 3, \dots$$

Since

$$v_n(x,0) = \sin\frac{n\pi x}{L},$$

v_n satisfies (12.1.1) with $f(x) = \sin n\pi x/L$. More generally, if $\alpha_1, \dots, \alpha_m$ are constants and

$$u_m(x,t) = \sum_{n=1}^{m} \alpha_n e^{-n^2\pi^2a^2t/L^2}\sin\frac{n\pi x}{L},$$

then u_m satisfies (12.1.1) with

$$f(x) = \sum_{n=1}^{m} \alpha_n \sin\frac{n\pi x}{L}.$$

This motivates the next definition.

Definition 12.1.1 The *formal solution* of the initial-boundary value problem

$$\begin{aligned} u_t &= a^2 u_{xx}, \quad 0 < x < L, \quad t > 0, \\ u(0,t) &= 0, \quad u(L,t) = 0, \quad t > 0, \\ u(x,0) &= f(x), \quad 0 \le x \le L \end{aligned} \tag{12.1.4}$$

is

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n e^{-n^2\pi^2 a^2 t/L^2} \sin \frac{n\pi x}{L}, \tag{12.1.5}$$

where

$$S(x) = \sum_{n=1}^{\infty} \alpha_n \sin \frac{n\pi x}{L}$$

is the Fourier sine series of f on $[0, L]$; that is,

$$\alpha_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx.$$

We use the term "formal solution" in this definition because it's not in general true that the infinite series in (12.1.5) actually satisfies all the requirements of the initial-boundary value problem (12.1.4) when it does, we say that it's an *actual solution* of (12.1.4).

Because of the negative exponentials in (12.1.5), u converges for all (x,t) with $t > 0$ (Exercise 54). Since each term in (12.1.5) satisfies the heat equation and the boundary conditions in (12.1.4), u also has these properties if u_t and u_{xx} can be obtained by differentiating the series in (12.1.5) term by term once with respect to t and twice with respect to x, for $t > 0$. However, it's not always legitimate to differentiate an infinite series term by term. The next theorem gives a useful sufficient condition for legitimacy of term by term differentiation of an infinite series. We omit the proof.

Theorem 12.1.2 *A convergent infinite series*

$$W(z) = \sum_{n=1}^{\infty} w_n(z)$$

can be differentiated term by term on a closed interval $[z_1, z_2]$ *to obtain*

$$W'(z) = \sum_{n=1}^{\infty} w_n'(z)$$

(where the derivatives at $z = z_1$ *and* $z = z_2$ *are one-sided) provided that* w_n' *is continuous on* $[z_1, z_2]$ *and*

$$|w_n'(z)| \le M_n, \quad z_1 \le z \le z_2, \quad n = 1, 2, 3, \dots,$$

where $M_1, M_2, \dots, M_n, \dots,$ *are constants such that the series* $\sum_{n=1}^{\infty} M_n$ *converges.*

Theorem 12.1.2, applied twice with $z = x$ and once with $z = t$, shows that u_{xx} and u_t can be obtained by differentiating u term by term if $t > 0$ (Exercise 54). Therefore u satisfies the heat equation and the boundary conditions in (12.1.4) for $t > 0$. Therefore, since $u(x, 0) = S(x)$ for $0 \le x \le L$, u is an actual solution of (12.1.4) if and only if $S(x) = f(x)$ for $0 \le x \le L$. From Theorem 11.3.2, this is true if f is continuous and piecewise smooth on $[0, L]$, and $f(0) = f(L) = 0$.

In this chapter we'll define formal solutions of several kinds of problems. When we ask you to solve such problems, we always mean that you should find a formal solution.

Example 12.1.1 Solve (12.1.4) with $f(x) = x(x^2 - 3Lx + 2L^2)$.

Solution From Example 11.3.6, the Fourier sine series of f on $[0, L]$ is

$$S(x) = \frac{12L^3}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin \frac{n\pi x}{L}.$$

Therefore

$$u(x,t) = \frac{12L^3}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} e^{-n^2\pi^2 a^2 t/L^2} \sin \frac{n\pi x}{L}. \qquad \blacksquare$$

If both ends of bar are insulated so that no heat can pass through them, then the boundary conditions are

$$u_x(0,t) = 0, \quad u_x(L,t) = 0, \quad t > 0.$$

We leave it to you (Exercise 1) to use the method of separation of variables and Theorem 11.1.3 to motivate the next definition.

Definition 12.1.3 The formal solution of the initial-boundary value problem

$$\begin{gathered} u_t = a^2 u_{xx}, \quad 0 < x < L, \quad t > 0, \\ u_x(0,t) = 0, \quad u_x(L,t) = 0, \quad t > 0, \\ u(x,0) = f(x), \quad 0 \le x \le L \end{gathered} \tag{12.1.6}$$

is

$$u(x,t) = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n e^{-n^2\pi^2 a^2 t/L^2} \cos \frac{n\pi x}{L},$$

where

$$C(x) = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos \frac{n\pi x}{L}$$

is the Fourier cosine series of f on $[0, L]$; that is,

$$\alpha_0 = \frac{1}{L} \int_0^L f(x)\, dx \quad \text{and} \quad \alpha_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx, \quad n = 1, 2, 3, \dots.$$

Example 12.1.2 Solve (12.1.6) with $f(x) = x$.

Solution From Example 11.3.1, the Fourier cosine series of f on $[0, L]$ is

$$C(x) = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{L}.$$

Therefore

$$u(x,t) = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} e^{-(2n-1)^2\pi^2 a^2 t/L^2} \cos \frac{(2n-1)\pi x}{L}. \qquad \blacksquare$$

We leave it to you (Exercise 2) to use the method of separation of variables and Theorem 11.1.4 to motivate the next definition.

Definition 12.1.4 The formal solution of the initial-boundary value problem

$$\begin{array}{c} u_t = a^2 u_{xx}, \quad 0 < x < L, \quad t > 0, \\ u(0,t) = 0, \quad u_x(L,t) = 0, \quad t > 0, \\ u(x,0) = f(x), \quad 0 \le x \le L \end{array} \tag{12.1.7}$$

is

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n e^{-(2n-1)^2\pi^2 a^2 t/4L^2} \sin \frac{(2n-1)\pi x}{2L},$$

where

$$S_M(x) = \sum_{n=1}^{\infty} \alpha_n \sin \frac{(2n-1)\pi x}{2L}$$

is the mixed Fourier sine series of f on $[0, L]$; that is,

$$\alpha_n = \frac{2}{L} \int_0^L f(x) \sin \frac{(2n-1)\pi x}{2L}\, dx.$$

Example 12.1.3 Solve (12.1.7) with $f(x) = x$.

Solution From Example 11.3.4, the mixed Fourier sine series of f on $[0, L]$ is

$$S_M(x) = -\frac{8L}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{2L}.$$

Therefore

$$u(x,t) = -\frac{8L}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} e^{-(2n-1)^2\pi^2 a^2 t/4L^2} \sin \frac{(2n-1)\pi x}{2L}. \quad \blacksquare$$

Figure 12.1.2 shows a graph of $u = u(x,t)$ plotted with respect to x for various values of t. The line $y = x$ corresponds to $t = 0$. The other curves correspond to positive values of t. As t increases, the graphs approach the line $u = 0$.

We leave it to you (Exercise 3) to use the method of separation of variables and Theorem 11.1.5 to motivate the next definition.

Definition 12.1.5 The formal solution of the initial-boundary value problem

$$\begin{array}{c} u_t = a^2 u_{xx}, \quad 0 < x < L, \quad t > 0, \\ u_x(0,t) = 0, \quad u(L,t) = 0, \quad t > 0, \\ u(x,0) = f(x), \quad 0 \le x \le L \end{array} \tag{12.1.8}$$

is

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n e^{-(2n-1)^2\pi^2 a^2 t/4L^2} \cos \frac{(2n-1)\pi x}{2L},$$

where

$$C_M(x) = \sum_{n=1}^{\infty} \alpha_n \cos \frac{(2n-1)\pi x}{2L}$$

is the mixed Fourier cosine series of f on $[0, L]$; that is,

$$\alpha_n = \frac{2}{L} \int_0^L f(x) \cos \frac{(2n-1)\pi x}{2L}\, dx.$$

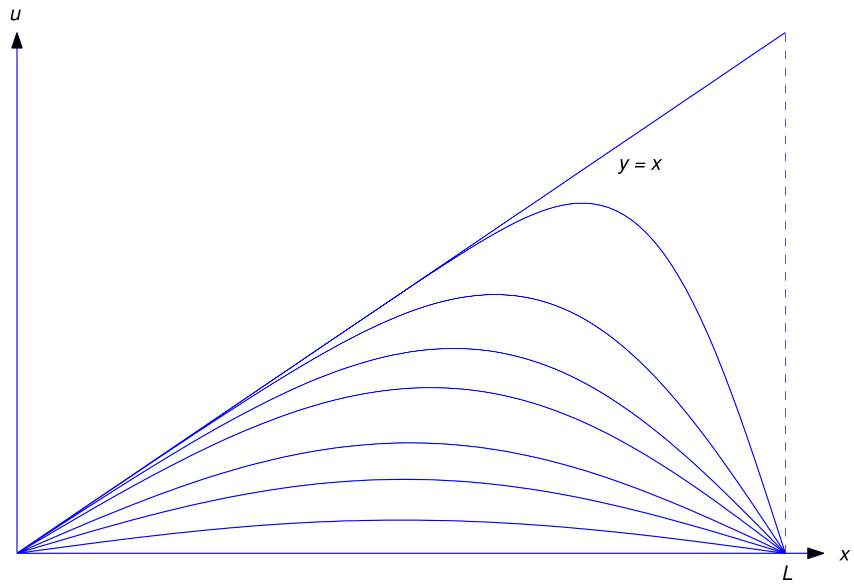

Figure 12.1.2

Example 12.1.4 Solve (12.1.8) with $f(x) = x - L$.

Solution From Example 11.3.3, the mixed Fourier cosine series of f on $[0, L]$ is

$$C_M(x) = -\frac{8L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{2L}.$$

Therefore

$$u(x,t) = -\frac{8L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}e^{-(2n-1)^2\pi^2a^2t/4L^2}\cos\frac{(2n-1)\pi x}{2L}.$$

Nonhomogeneous Problems

A problem of the form

$$\begin{gathered} u_t = a^2u_{xx} + h(x), \quad 0 < x < L, \quad t > 0, \\ u(0,t) = u_0, \quad u(L,t) = u_L, \quad t > 0, \\ u(x,0) = f(x), \quad 0 \le x \le L \end{gathered} \tag{12.1.9}$$

can be transformed to a problem that can be solved by separation of variables. We write

$$u(x,t) = v(x,t) + q(x), \tag{12.1.10}$$

where q is to be determined. Then

$$u_t = v_t \quad \text{and} \quad u_{xx} = v_{xx} + q''$$

so u satisfies (12.1.9) if v satisfies

$$\begin{gathered} v_t = a^2 v_{xx} + a^2 q''(x) + h(x), \quad 0 < x < L, \quad t > 0, \\ v(0,t) = u_0 - q(0), \quad v(L,t) = u_L - q(L), \quad t > 0, \\ v(x,0) = f(x) - q(x), \quad 0 \le x \le L. \end{gathered}$$

This reduces to

$$\begin{gathered} v_t = a^2 v_{xx}, \quad 0 < x < L, \quad t > 0, \\ v(0,t) = 0, \quad v(L,t) = 0, \quad t > 0, \\ v(x,0) = f(x) - q(x), \quad 0 \le x \le L \end{gathered} \tag{12.1.11}$$

if

$$a^2 q'' + h(x) = 0, \quad q(0) = u_0, \quad q(L) = u_L.$$

We can obtain q by integrating $q'' = -h/a^2$ twice and choosing the constants of integration so that $q(0) = u_0$ and $q(L) = u_L$. Then we can solve (12.1.11) for v by separation of variables, and (12.1.10) is the solution of (12.1.9).

Example 12.1.5 Solve

$$\begin{gathered} u_t = u_{xx} - 2, \quad 0 < x < 1, \quad t > 0, \\ u(0,t) = -1, \quad u(1,t) = 1, \quad t > 0, \\ u(x,0) = x^3 - 2x^2 + 3x - 1, \quad 0 \le x \le 1. \end{gathered}$$

Solution We leave it to you to show that

$$q(x) = x^2 + x - 1$$

satisfies

$$q'' - 2 = 0, \quad q(0) = -1, \quad q(1) = 1.$$

Therefore

$$u(x,t) = v(x,t) + x^2 + x - 1,$$

where

$$v_t = v_{xx}, \quad 0 < x < 1, \quad t > 0,$$

$$v(0,t) = 0, \quad v(1,t) = 0, \quad t > 0,$$

and

$$v(x,0) = x^3 - 2x^2 + 3x - 1 - x^2 - x + 1 = x(x^2 - 3x + 2).$$

From Example 12.1.1 with $a = 1$ and $L = 1$,

$$v(x,t) = \frac{12}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} e^{-n^2\pi^2 t} \sin n\pi x.$$

Therefore

$$u(x,t) = x^2 + x - 1 + \frac{12}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} e^{-n^2\pi^2 t} \sin n\pi x. \qquad \blacksquare$$

A similar procedure works if the boundary conditions in (12.1.11) are replaced by mixed boundary conditions

$$u_x(0,t) = u_0, \quad u(L,t) = u_L, \quad t > 0$$

or

$$u(0,t) = u_0, \quad u_x(L,t) = u_L, \quad t > 0;$$

however, this isn't true in general for the boundary conditions

$$u_x(0,t) = u_0, \quad u_x(L,t) = u_L, \quad t > 0.$$

(See Exercise 47.)

USING TECHNOLOGY

Numerical experiments can enhance your understanding of the solutions of initial-boundary value problems. To be specific, consider the formal solution

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n e^{-n^2\pi^2a^2t/L^2} \sin \frac{n\pi x}{L},$$

of (12.1.4), where

$$S(x) = \sum_{n=1}^{\infty} \alpha_n \sin \frac{n\pi x}{L}$$

is the Fourier sine series of f on $[0, L]$. Consider the m-th partial sum

$$u_m(x,t) = \sum_{n=1}^{m} \alpha_n e^{-n^2\pi^2a^2t/L^2} \sin \frac{n\pi x}{L}. \tag{12.1.12}$$

For several fixed values of t (including $t = 0$), graph $u_m(x,t)$ versus t. In some cases it may be useful to graph the curves corresponding to the various values of t on the same axes in other cases you may want to graph the various curves sucessively (for increasing values of t), and create a primitive motion picture on your monitor. Repeat this experiment for several values of m, to compare how the results depend upon m for small and large values of t. However, keep in mind that the meanings of "small" and "large" in this case depend upon the constants a^2 and L^2. A good way to handle this is to rewrite (12.1.12) as

$$u_m(x,t) = \sum_{n=1}^{m} \alpha_n e^{-n^2\tau} \sin \frac{n\pi x}{L},$$

where

$$\tau = \frac{\pi^2a^2t}{L^2}, \tag{12.1.13}$$

and graph u_m versus x for selected values of τ.

These comments also apply to the situations considered in Definitions 12.1.3-12.1.5, except that (12.1.13) should be replaced by

$$\tau = \frac{\pi^2a^2t}{4L^2},$$

in Definitions 12.1.4 and 12.1.5.

In some of the exercises we say "perform numerical experiments." This means that you should perform the computations just described with the formal solution obtained in the exercise.

12.1 Exercises

1. Explain Definition 12.1.3.

2. Explain Definition 12.1.4.

3. Explain Definition 12.1.5.

4. $\boxed{\text{C}}$ Perform numerical experiments with the formal solution obtained in Example 12.1.1.

5. $\boxed{\text{C}}$ Perform numerical experiments with the formal solution obtained in Example 12.1.2.

6. $\boxed{\text{C}}$ Perform numerical experiments with the formal solution obtained in Example 12.1.3.

7. $\boxed{\text{C}}$ Perform numerical experiments with the formal solution obtained in Example 12.1.4.

In Exercises 8-42 solve the initial-boundary value problem. Where indicated by $\boxed{C}$ *, perform numerical experiments. To simplify the computation of coefficients in some of these problems, check first to see if* $u(x,0)$ *is a polynomial that satisfies the boundary conditions. If it does, apply Theorem 11.3.5; also, see Exercises* 11.3.35**(b)**, 11.3.42**(b)**, *and* 11.3.50**(b)**.

8. $u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = x(1-x), \quad 0 \le x \le 1$

9. $u_t = 9u_{xx}, \quad 0 < x < 4, \quad t > 0,$
$u(0,t) = 0, \quad u(4,t) = 0, \quad t > 0,$
$u(x,0) = 1, \quad 0 \le x \le 4$

10. $u_t = 3u_{xx}, \quad 0 < x < \pi, \quad t > 0,$
$u(0,t) = 0, \quad u(\pi,t) = 0, \quad t > 0,$
$u(x,0) = x\sin x, \quad 0 \le x \le \pi$

11. $\boxed{\text{C}}$ $u_t = 9u_{xx}, \quad 0 < x < 2, \quad t > 0,$
$u(0,t) = 0, \quad u(2,t) = 0, \quad t > 0,$
$u(x,0) = x^2(2-x), \quad 0 \le x \le 2$

12. $u_t = 4u_{xx}, \quad 0 < x < 3, \quad t > 0,$
$u(0,t) = 0, \quad u(3,t) = 0, \quad t > 0,$
$u(x,0) = x(9-x^2), \quad 0 \le x \le 3$

13. $u_t = 4u_{xx}, \quad 0 < x < 2, \quad t > 0,$
$u(0,t) = 0, \quad u(2,t) = 0, \quad t > 0,$
$$u(x,0) = \left\{ \begin{array}{cl} x, & 0 \le x \le 1, \\ 2-x, & 1 \le x \le 2. \end{array} \right.$$

14. $u_t = 7u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = x(3x^4 - 10x^2 + 7), \quad 0 \le x \le 1$

15. $u_t = 5u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = x(x^3 - 2x^2 + 1), \quad 0 \le x \le 1$

16. $u_t = 2u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = x(3x^4 - 5x^3 + 2), \quad 0 \le x \le 1$

17. $\boxed{\text{C}}$ $u_t = 9u_{xx}, \quad 0 < x < 4, \quad t > 0,$
$u_x(0,t) = 0, \quad u_x(4,t) = 0, \quad t > 0,$
$u(x,0) = x^2, \quad 0 \le x \le 4$

18. $u_t = 4u_{xx}, \quad 0 < x < 2, \quad t > 0,$
$u_x(0,t) = 0, \quad u_x(2,t) = 0, \quad t > 0,$
$u(x,0) = x(x-4), \quad 0 \le x \le 2$

19. $\boxed{\text{C}}$ $u_t = 9u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u_x(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0,$
$u(x,0) = x(1-x), \quad 0 \le x \le 1$

20. $u_t = 3u_{xx}, \quad 0 < x < 2, \quad t > 0,$
$u_x(0,t) = 0, \quad u_x(2,t) = 0, \quad t > 0,$
$u(x,0) = 2x^2(3-x), \quad 0 \le x \le 2$

21. $u_t = 5u_{xx}, \quad 0 < x < \sqrt{2}, \quad t > 0,$
$u_x(0,t) = 0, \quad u_x(\sqrt{2},t) = 0, \quad t > 0,$
$u(x,0) = 3x^2(x^2-4), \quad 0 \le x \le \sqrt{2}$

22. $\boxed{\text{C}}$ $u_t = 3u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u_x(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0,$
$u(x,0) = x^3(3x-4), \quad 0 \le x \le 1$

23. $u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u_x(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0,$
$u(x,0) = x^2(3x^2-8x+6), \quad 0 \le x \le 1$

24. $u_t = u_{xx}, \quad 0 < x < \pi, \quad t > 0,$
$u_x(0,t) = 0, \quad u_x(\pi,t) = 0, \quad t > 0,$
$u(x,0) = x^2(x-\pi)^2, \quad 0 \le x \le \pi$

25. $u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0,$
$u(x,0) = \sin \pi x, \quad 0 \le x \le 1$

26. $\boxed{\text{C}}$ $u_t = 3u_{xx}, \quad 0 < x < \pi, \quad t > 0,$
$u(0,t) = 0, \quad u_x(\pi,t) = 0, \quad t > 0,$
$u(x,0) = x(\pi - x), \quad 0 \le x \le \pi$

27. $u_t = 5u_{xx}, \quad 0 < x < 2, \quad t > 0,$
$u(0,t) = 0, \quad u_x(2,t) = 0, \quad t > 0,$
$u(x,0) = x(4-x), \quad 0 \le x \le 2$

28. $u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0,$
$u(x,0) = x^2(3-2x), \quad 0 \le x \le 1$

29. $u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0,$
$u(x,0) = (x-1)^3 + 1, \quad 0 \le x \le 1$

30. $\boxed{\text{C}}$ $u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0,$
$u(x,0) = x(x^2-3), \quad 0 \le x \le 1$

31. $u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0,$
$u(x,0) = x^3(3x-4), \quad 0 \le x \le 1$

32. $u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0,$
$u(x,0) = x(x^3 - 2x^2 + 2), \quad 0 \le x \le 1$

33. $u_t = 3u_{xx}, \quad 0 < x < \pi, \quad t > 0,$
$u_x(0,t) = 0, \quad u(\pi,t) = 0, \quad t > 0,$
$u(x,0) = x^2(\pi - x), \quad 0 \le x \le \pi$

34. $u_t = 16u_{xx}, \quad 0 < x < 2\pi, \quad t > 0,$
$u_x(0,t) = 0, \quad u(2\pi,t) = 0, \quad t > 0,$
$u(x,0) = 4, \quad 0 \le x \le 2\pi$

35. $u_t = 9u_{xx}, \quad 0 < x < 4, \quad t > 0,$
$u_x(0,t) = 0, \quad u(4,t) = 0, \quad t > 0,$
$u(x,0) = x^2, \quad 0 \le x \le 4$

36. $\boxed{\text{C}}$ $u_t = 3u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u_x(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = 1 - x, \quad 0 \le x \le 1$

37. $u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u_x(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = 1 - x^3, \quad 0 \le x \le 1$

38. $u_t = 7u_{xx}, \quad 0 < x < \pi, \quad t > 0,$
$u_x(0,t) = 0, \quad u(\pi,t) = 0, \quad t > 0,$
$u(x,0) = \pi^2 - x^2, \quad 0 \le x \le \pi$

39. $u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u_x(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = 4x^3 + 3x^2 - 7, \quad 0 \le x \le 1$

40. $u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u_x(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = 2x^3 + 3x^2 - 5, \quad 0 \le x \le 1$

41. $\boxed{\text{C}}$ $u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u_x(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = x^4 - 4x^3 + 6x^2 - 3, \quad 0 \le x \le 1$

42. $u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u_x(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = x^4 - 2x^3 + 1, \quad 0 \le x \le 1$

In Exercises 43-46 solve the initial-boundary value problem. Perform numerical experiments for specific values of L and a.

43. $\boxed{\text{C}}$ $u_t = a^2u_{xx}, \quad 0 < x < L, \quad t > 0,$
$u_x(0,t) = 0, \quad u_x(L,t) = 0, \quad t > 0,$
$$u(x,0) = \begin{cases} 1, & 0 \le x \le \frac{L}{2}, \\ 0, & \frac{L}{2} < x < L \end{cases}$$

44. $\boxed{\text{C}}$ $u_t = a^2u_{xx}, \quad 0 < x < L, \quad t > 0,$
$u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0,$
$$u(x,0) = \begin{cases} 1, & 0 \le x \le \frac{L}{2}, \\ 0, & \frac{L}{2} < x < L \end{cases}$$