Instructions. Answer each of the questions on your own paper, and be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper.

1. [**10 Points**] Let $A = \begin{bmatrix} 8 & x \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$. Determine all values of x for which $AB = BA$.

   ▶ **Solution.** $AB = \begin{bmatrix} 8 & x \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 8+x & 16 \\ 8 & 6 \end{bmatrix}$ and $BA = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 8 & x \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 14 & x+10 \\ 8 & x \end{bmatrix}$. These matrices are equal if $8+x = 14$, $16 = x+10$, $8 = 8$, and $6 = x$. These are all satisfied for $\boxed{x = 6}$. ◀

2. [**5 Points**] Let A be a 3×1 matrix, B a 2×3 matrix, and C a 2×1 matrix. which of the following matrix products is well defined and the result is a 1×2 matrix?

   (a) AB

   (b) B^TC

   (c) C^TBA

   (d) BAC^T

   (e) $\boxed{(BA)^T}$ **Answer.**

3. [**20 Points**] Find all solutions of the following system of linear equations. Be sure to show all your steps!

$$\begin{array}{ccccccccc} x_1 & + & 2x_2 & + & 3x_3 & + & x_4 & = & 8 \\ x_1 & + & 3x_2 & & & + & x_4 & = & 7 \\ x_1 & & & + & 2x_3 & + & x_4 & = & 3 \end{array}$$

   ▶ **Solution.** Use Gauss-Jordan elimination on the augmented matrix A:

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 & 8 \\ 1 & 3 & 0 & 1 & 7 \\ 1 & 0 & 2 & 1 & 3 \end{bmatrix} \begin{array}{c} R_2 \mapsto R_2 - R_1 \\ \longrightarrow \\ R_3 \mapsto R_3 - R_1 \end{array} \begin{bmatrix} 1 & 2 & 3 & 1 & 8 \\ 0 & 1 & -3 & 0 & -1 \\ 0 & -2 & -1 & 0 & -5 \end{bmatrix}$$

$$\begin{array}{c} \\ \longrightarrow \\ R_3 \mapsto R_3 + 2R_2 \end{array} \begin{bmatrix} 1 & 2 & 3 & 1 & 8 \\ 0 & 1 & -3 & 0 & -1 \\ 0 & 0 & -7 & 0 & -7 \end{bmatrix} \begin{array}{c} -\frac{1}{7}R_2 \\ \longrightarrow \\ \end{array} \begin{bmatrix} 1 & 2 & 3 & 1 & 8 \\ 0 & 1 & -3 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{c} R_2 \mapsto R_2 + 3R_3 \\ \longrightarrow \\ R_1 \mapsto R_1 - 3R_3 \end{array} \begin{bmatrix} 1 & 2 & 0 & 1 & 5 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{array}{c} R_1 \mapsto R_1 - 2R_2 \\ \longrightarrow \\ \end{array} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

The last matrix is in reduced row-echelon form and is the augmented matrix of the linear system

$$\begin{array}{rcrcrcrcl} x_1 & & & & & + & x_4 & = & 1 \\ & & x_2 & & & & & = & 2 \\ & & & & x_3 & & & = & 1 \end{array}$$

The solution set of this last system is the same as that of the original system and is given by:

$$\mathcal{S} = \{(1 - x_4,\, 2,\, 1,\, x_4) :\; x_4 \text{ is arbitrary}\}.$$

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4. [**20 Points**] Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$.

   (a) Compute A^{-1}.

   ▶ **Solution.** Use Gauss-Jordan elimination of the augmented matrix $\begin{bmatrix} A & | & I_3 \end{bmatrix}$:

$$\begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \mapsto R_2 - 2R_1} \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 2 & 1 & -2 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 \mapsto R_3 - 2R_2} \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 3 & -2 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{3}R_3} \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix} \xrightarrow{R_2 \mapsto R_2 + R_3} \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

$$\xrightarrow{R_1 \mapsto R_1 + R_2} \begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

   Hence,

$$A^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

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   (b) Using your answer to part (a), solve the system of equations

$$\begin{array}{rcrcrcl} x_1 & - & x_2 & & & = & 1 \\ 2x_1 & & & + & x_3 & = & 1 \\ & & x_2 & - & x_3 & = & 1 \end{array}$$

▶ **Solution.**

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

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5. [**15 Points**] Find conditions that b_1, b_2, and b_3 must satisfy for the following system to be consistent:

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & - & 3x_3 & = & b_1 \\ 2x_1 & + & 3x_2 & + & 3x_3 & = & b_2 \\ 5x_1 & + & 9x_2 & - & 6x_3 & = & b_3 \end{array}$$

▶ **Solution.** Start by trying to row reduce the augmented matrix:

$$\begin{bmatrix} 1 & 2 & -3 & b_1 \\ 2 & 3 & 3 & b_2 \\ 5 & 9 & -6 & b_3 \end{bmatrix} \begin{array}{c} R_2 \mapsto R_2 - 2R_1 \\ \longrightarrow \\ R_3 \mapsto R_3 - 5R_1 \end{array} \begin{bmatrix} 1 & 2 & -3 & b_1 \\ 0 & -1 & 9 & b_2 - 2b_1 \\ 0 & -1 & 9 & b_3 - 5b_1 \end{bmatrix}$$

$$\begin{array}{c} -R_2 \\ \longrightarrow \\ R_3 \mapsto R_3 + R_2 \end{array} \begin{bmatrix} 1 & 2 & -3 & b_1 \\ 0 & 1 & 9 & -b_2 + 2b_1 \\ 0 & 0 & 0 & b_3 - 3b_1 - b_2 \end{bmatrix}$$

In order for the system to be consistent, the last entry in the last column must be 0, and this entry being 0 is also sufficient to be able to complete the row reduction and solve the system. Hence, the system is consistent if and only if $\boxed{3b_1 + b_2 = b_3}$. ◀

6. [**10 Points**] Find all the values of y for which the matrix

$$A = \begin{bmatrix} 2 & 1 & 3y & 4 \\ 0 & y-1 & 4 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & y & 4 \end{bmatrix}$$

is **not** invertible.

▶ **Solution.** The matrix A is not invertible if and only if $\det A = 0$. But using cofactor expansions along the first column gives:

$$\det A = 2 \begin{vmatrix} y-1 & 4 & 0 \\ 0 & 2 & 1 \\ 0 & y & 4 \end{vmatrix} = 2(y-1) \begin{vmatrix} 2 & 1 \\ y & 4 \end{vmatrix} = 2(y-1)(8-y).$$

Thus, $\det A = 0$ if and only if $y = 1$ or $y = 8$. ◀

7. [**10 Points**] Let A be an $n \times n$ invertible matrix. Determine whether each of the following statements is True (**T**) or False (**F**)?

(a) $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions. **F**

(b) A is a product of elementary matrices. **T**

(c) $\det A = 0$. **F**

(d) The reduced row echelon form of A is the identity matrix I_n. **T**

(e) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ matrix $\mathbf{b}$. **T**

8. [**10 Points**] Suppose that A is the 3×3 matrix

$$\begin{bmatrix} 1 & 8 & 3 \\ x & y & z \\ -3 & 7 & 2 \end{bmatrix}.$$

Assuming that $\det(A) = 5$, compute the determinant of the following matrices:

$$\text{(a) } B = 2A^2; \qquad \text{(b) } C = \begin{bmatrix} x & y & z \\ 1 & 8 & 3 \\ 6+4x & -14+4y & -4+4z \end{bmatrix}.$$

► **Solution.** (a) $\det B = 2^3 \det A^2 = 2^3(\det A)^2 = 8 \times 5^2 = 200$.

(b) C is obtained from A by the following sequence of elementary row operations:

$$-2R_3; \qquad R_3 \mapsto R_3 + 4R_1; \qquad R_1 \leftrightarrow R_2.$$

The first operation multiplies the determinant by -2, the second does not change the determinant, and the third multiplies the determinant by -1. Thus, the determinant of the final matrix C is given by:

$$\det C = (-2)(-1)\det A = 10.$$

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