Instructions. Answer each of the questions on your own paper, and be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper.

1. [**12 Points**] Which of the following subsets of $\mathbb{R}^3$ are subspaces. Give a reason in each case.

   (a) $S_1 = \{(a,\, b,\, c) : a + b + c = 0\}$.

   ▶ **Solution.** Let $\mathbf{v}_1 = (a_1,\, b_1,\, c_1)$ and $\mathbf{v}_2 = (a_2,\, b_2,\, c_2)$ be two arbitrary elements of S_1. This means that $a_1 + b_1 + c_1 = 0$ and $a_2 + b_2 + c_2 = 0$. Then, $\mathbf{v}_1 + \mathbf{v}_2 = (a_1 + a_2,\, b_1 + b_2,\, c_1 + c_2)$ and $(a_1 + a_2) + (b_1 + b_2) + (c_1 + c_2) = (a_1 + b_1 + c_1) + (a_2 + b_2 + c_2) = 0 + 0 = 0$ so $\mathbf{v}_1 + \mathbf{v}_2 \in S_1$. Also, if $k \in \mathbb{R}$ is an arbitrary scalar, then $k\mathbf{v}_1 = (ka_1,\, kb_1,\, kc_1)$ and $ka_1 + kb_1 + kc_1 = k(a_1 + b_1 + c_1) = k0 = 0$. Thus, S_1 is closed under vector addition and closed under scalar multiplication, so it is a **subspace**. ◀

   (b) $S_2 = \{(a,\, b,\, c) : abc = 0\}$.

   ▶ **Solution.** Let $\mathbf{v}_1 = (1,\, 1,\, 0)$ and $\mathbf{v}_2 = (0,\, 0,\, 1)$. Then $\mathbf{v}_1$ and $\mathbf{v}_2$ are in S_2, but $\mathbf{v}_1 + \mathbf{v}_2 = (1,\, 1,\, 1)$ is not in S_2. Hence, S_1 is not closed under vector addition, so it is **not a subspace**. ◀

2. [**16 Points**] Find a basis for and the dimension of the solution space of the homogeneous system

$$\begin{array}{rcrcrcrcl} x_1 & - & 3x_2 & & & - & 5x_4 & = & 0 \\ -x_1 & + & 4x_2 & + & x_3 & + & 7x_4 & = & 0 \\ 2x_1 & + & x_2 & + & 7x_3 & + & 4x_4 & = & 0 \\ 2x_1 & - & 2x_2 & + & 4x_3 & - & 2x_4 & = & 0 \end{array}$$

You may use the fact that the coefficient matrix of this system is the matrix $A = \begin{bmatrix} 1 & -3 & 0 & -5 \\ -1 & 4 & 1 & 7 \\ 2 & 1 & 7 & 4 \\ 2 & -2 & 4 & -2 \end{bmatrix}$ and the matrix $R = \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is the reduced row-echelon matrix associated to A.

▶ **Solution.** Since R is the reduced row-echelon matrix associated to A, the given system of equations is equivalent to the system with coefficient matrix R:

$$\begin{array}{rcrcrcrcl} x_1 & & & + & 3x_3 & & x_4 & = & 0 \\ & & x_2 & + & x_3 & + & 2x_4 & = & 0 \\ & & & & & & 0 & = & 0 \\ & & & & & & 0 & = & 0 \end{array}$$

This equation has free variables x_3 and x_4 so we let $x_3 = s$ and $x_4 = t$. Then the solution set is

$$\mathcal{S} = \{(-3s - t,\, -s - t,\, s,\, t) : \ s \text{ and } t \text{ are arbitrary}\}\,.$$

Writing,
$$(-3s-t,\ -s-t,\ s,\ t) = s(-3,\ -1,\ 1,\ 0) + t(-1,\ -1,\ 0,\ 1),$$
we see that every vector $\mathbf{v} = (-3s-t,\ -s-t,\ s,\ t) \in \mathcal{S}$ can be written as a linear combination $\mathbf{v} = s\mathbf{v}_1 + t\mathbf{v}_2$ where $\mathbf{v}_1 = (-3,\ -1,\ 1,\ 0)$ and $\mathbf{v}_2 = (-1,\ -1,\ 0,\ 1)$. Thus, $\mathcal{S} = \operatorname{Span}\{\mathbf{v}_1,\ \mathbf{v}_2\}$ and since $\mathbf{v}_1 \neq k\mathbf{v}_2$ and $\mathbf{v}_2 \neq k\mathbf{v}_1$ for any scalar k, it follows that $\{\mathbf{v}_1,\ \mathbf{v}_2\}$ is also linearly independent. Thus $\{\mathbf{v}_1,\ \mathbf{v}_2\}$ is a basis of $\mathcal{S}$. Since there are two vectors in this basis, it follows that the dimension of $\mathcal{S}$ is 2. ◀

3. [**16 Points**] Let $\mathbf{v}_1 = (-1,\ 2,\ 3,\ 1)$, $\mathbf{v}_2 = (5,\ 0,\ 2,\ -3)$, and $\mathbf{v}_3 = (3,\ 4,\ 8,\ 1)$. Determine if the set $S = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\} \subseteq \mathbb{R}^4$ is linearly independent. If it is not linearly independent, find an explicit liner combination $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$ with at least one scalar $k_i \neq 0$.

▶ **Solution.** We need to find all of the solutions of the vector equation
$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}.$$
Writing out this equation in terms of components gives a system of 4 equations in 3 unknowns:
$$\begin{array}{rcrcrcl} -k_1 & + & 5k_2 & + & 3k_3 & = & 0 \\ 2k_1 & & & + & 4k_3 & = & 0 \\ 3k_3 & + & 2k_2 & + & 8k_3 & = & 0 \\ k_1 & - & 3k_2 & + & k_3 & = & 0. \end{array}$$
This system has coefficient matrix
$$A = \begin{bmatrix} -1 & 5 & 3 \\ 2 & 0 & 4 \\ 3 & 2 & 8 \\ 1 & -3 & 1 \end{bmatrix}.$$
After a few steps, this matrix row reduces to
$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$
This corresponds to the system $k_1 = 0$, $k_2 = 0$, $k_3 = 0$, which means that the vectors are **linearly independent**. ◀

4. [**16 Points**] Let $W = \operatorname{Span}\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\}$ where $\mathbf{v}_1 = (1,\ -1,\ 2)$, $\mathbf{v}_2 = (3,\ -2,\ 5)$, and $\mathbf{v}_3 = (2,\ -1,\ 3)$.

   (a) Find a subset $S \subset \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\}$ that forms a basis of W. What is the dimension of W?

▶ **Solution.** First find any dependency relations among the given vectors. Solving the vector equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$ leads to a system of linear equations with coefficient matrix

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 \\ -1 & -2 & -1 \\ 2 & 5 & 3 \end{bmatrix}.$$

Row reduce A to get

$$A \mapsto \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, the solutions of the vector equation are $(k_1, k_2, k_3) = (k_3, -k_3, k_3) = k_3(1, -1, 1)$. Hence the three vectors are linearly dependent. In fact, $\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$. Thus, $\mathbf{v}_3 = \mathbf{v}_2 - \mathbf{v}_1 \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. We conclude that $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Since neither $\mathbf{v}_1$ or $\mathbf{v}_2$ is a scalar multiple of the other, it follows that the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis of W. Since there are 2 vectors in this basis, the dimension of W is 2. ◀

(b) Write each vector $\mathbf{v}_i$ that is not in S as a linear combination of the vectors in S.

▶ **Solution.** As shown in the previous part, $\mathbf{v}_3 = \mathbf{v}_2 - \mathbf{v}_1$. ◀

5. [**12 Points**] Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be the basis of $\mathbb{R}^3$ with $\mathbf{v}_1 = (1, 0, 0)$, $\mathbf{v}_2 = (2, 2, 0)$, and $\mathbf{v}_3 = (3, 3, 3)$. If $\mathbf{v} = (2, -1, 3)$ compute the coordinate vector $(\mathbf{v})_S$ of $\mathbf{v}$ relative to the basis S.

▶ **Solution.** $(\mathbf{v})_S = (c_1, c_2, c_3)$ where $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$. To find c_1, c_2, c_3 the vector equation

$$(2, -1, 3) = c_1(1, 0, 0) + c_2(2, 2, 0) + c_3(3, 3, 3)$$

leads to a system of linear equations

$$\begin{array}{ccccccc} c_1 & + & 2c_2 & + & 3c_3 & = & 2 \\ & & 2c_2 & + & 3c_3 & = & -1 \\ & & & & 3c_3 & = & 3. \end{array}$$

Solving this system gives $c_1 = 3$, $c_2 = -2$, $c_3 = 1$. Thus, $(\mathbf{v})_S = (3, -2, 1)$. ◀

6. [**12 Points**] Let $\mathbf{u} = (-3, 2, 1, 0)$ and $\mathbf{a} = (0, 2, -1, 1)$ be vectors in $\mathbb{R}^4$. Compute the following:

(a) The vector component $\text{proj}_{\mathbf{a}}\,\mathbf{u}$ of $\mathbf{u}$ along $\mathbf{a}$.

▶ **Solution.** $\operatorname{proj}_{\mathbf{a}} \mathbf{u} = \dfrac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = \dfrac{(-3)0 + 2\cdot 2 + 1(-1) + 0\cdot 1}{0^2 + 2^2 + (-1)^2 + 1^2}(0,\, 2,\, -1,\, 1) = \frac{1}{2}(0,\, 2,\, -1,\, 1) = (0,\, 1,\, -\frac{1}{2},\, \frac{1}{2})$ ◀

(b) The vector component of **u** orthogonal to **a**.

▶ **Solution.** The vector component of **u** orthogonal to **a** is

$$\mathbf{u} - \operatorname*{proj}_{\mathbf{a}} \mathbf{u} = (-3,\, 2,\, 1,\, 0) - (0,\, 1,\, -\frac{1}{2},\, \frac{1}{2}) = (-3,\, 1,\, \frac{3}{2},\, -\frac{1}{2}).$$

◀

(c) The cosine of the angle between **u** and **a**.

▶ **Solution.** If θ is the angle between **u** and **a**, then

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{u}\|\,\|\mathbf{a}\|} = \frac{3}{\sqrt{14}\sqrt{6}} = \frac{3}{2\sqrt{21}}.$$

◀

7. **[16 Points]** Complete the following definitions:

(a) A vector **w** is a *linear combination* of vectors $\{\mathbf{v}_1,\, \ldots,\, \mathbf{v}_p\}$ if …

▶ **Solution.** if $\mathbf{w} = k_1\mathbf{v}_1 + \cdots + k_p\mathbf{v}_p$ for some scalars $k_1,\, \ldots,\, k_p$. ◀

(b) A set $S = \{\mathbf{v}_1,\, \ldots,\, \mathbf{v}_p\}$ of vectors in a vector space V is *linearly dependent* if …

▶ **Solution.** $k_1\mathbf{v}_1 + \cdots + k_p\mathbf{v}_p = \mathbf{0}$ for some scalars $k_1,\, \ldots,\, k_p$ with at least one of the scalars $k_i \neq 0$. ◀

(c) A *basis* of a vector space V is a set $S = \{\mathbf{v}_1,\, \ldots,\, \mathbf{v}_p\}$ of vectors such that …

▶ **Solution.** S is linearly independent and spans V. ◀

(d) The *dimension* of a vector space V is …

▶ **Solution.** the number of vectors in a basis. ◀