Instructions. Answer each of the questions on your own paper, and be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper.

1. [**20 Points**] Let $A = \begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ -2 & -3 & 1 & -1 & -5 \\ -3 & -4 & 0 & -2 & -3 \\ 3 & 6 & -6 & 5 & 1 \end{bmatrix}$ and let $R = \begin{bmatrix} 1 & 0 & 4 & 0 & -3 \\ 0 & 1 & -3 & 0 & 5 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Then R is the reduced row-echelon matrix associated to A. You may assume this as given. *It is not necessary to check it.* Using this fact, answer the following questions.

(a) Compute a basis for the row space Row(A) of A.

▶ **Solution.** A basis for Row(A) consists of the nonzero rows of R. Thus,

$$S = \left\{ \begin{bmatrix} 1 & 0 & 4 & 0 & -3 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -3 & 0 & 5 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & -4 \end{bmatrix} \right\}$$

is a basis of Row(A). ◀

(b) Compute a basis for the column space Col(A) of A.

▶ **Solution.** A basis for the column space consists of the columns of A that correspond to the columns of R that contain the leading one's. Thus a basis is

$$S = \left\{ \begin{bmatrix} 1 \\ -2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -4 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \\ 5 \end{bmatrix} \right\}.$$

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(c) Compute a basis for the nullspace Null(A) of A.

▶ **Solution.** The nullspace of A is the same as the nullspace of the reduced row echelon matrix R. Thus, $\mathbf{x} \in \text{Null}(A)$ if and only if

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -4x_3 + 3x_5 \\ 3x_3 - 5x_5 \\ x_3 \\ 4x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ -5 \\ 0 \\ 4 \\ 1 \end{bmatrix}.$$

Hence, a basis of Null(A) is

$$\left\{ \begin{bmatrix} -4 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 0 \\ 4 \\ 1 \end{bmatrix} \right\}.$$

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(d) What is the rank of A?

▶ **Solution.** Rank$(A) = \dim \mathrm{Col}(A) = 3$. ◀

2. [**7 Points**] Is the vector $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ an eigenvector of the matrix $A = \begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}$? If so, what is the corresponding eigenvalue?

▶ **Solution.** $A \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, so the given vector is an eigenvector with eigenvalue -2. ◀

3. [**12 Points**] Determine the characteristic polynomial and the eigenvalues of the matrix

$$A = \begin{bmatrix} 4 & 1 & -1 \\ 0 & 5 & -6 \\ 0 & -1 & 4 \end{bmatrix}.$$

You are *not* asked to find any eigenvectors.

▶ **Solution.** The characteristic polynomial is

$$\begin{aligned} \det(\lambda I - A) &= \begin{bmatrix} \lambda - 4 & -1 & 1 \\ 0 & \lambda - 5 & 6 \\ 0 & 1 & \lambda - 4 \end{bmatrix} \\ &= (\lambda - 4) \det \begin{bmatrix} \lambda - 5 & 6 \\ 1 & \lambda - 4 \end{bmatrix} \\ &= (\lambda - 4)((\lambda - 5)(\lambda - 4) - 6) \\ &= (\lambda - 4)(\lambda^2 - 9\lambda + 14) \\ &= (\lambda - 4)(\lambda - 7)(\lambda - 2). \end{aligned}$$

Hence, the eigenvalues are 4, 7, and 2. ◀

4. [**16 Points**] Find an invertible matrix P and diagonal matrix D such that $P^{-1}AP = D$, where the matrix $A = \begin{bmatrix} 2 & 10 \\ 1 & -1 \end{bmatrix}$ and you may assume that the eigenvalues of A are $\lambda = 4$ and $\lambda = -3$.

▶ **Solution.** $D = \begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}$ and $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$ where $\mathbf{v}_2$ is an eigenvector with eigenvalue 4 and $\mathbf{v}_2$ is an eigenvector with eigenvalue -3. $A - 4I = \begin{bmatrix} -2 & 10 \\ 1 & -5 \end{bmatrix}$ which row reduces to $\begin{bmatrix} 1 & -5 \\ 0 & 0 \end{bmatrix}$ so that $\mathbf{v}_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, while $A + 3I = \begin{bmatrix} 5 & 10 \\ 1 & 2 \end{bmatrix}$ which row reduces to

$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ so that $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Hence $P = \begin{bmatrix} 5 & 2 \\ 1 & -1 \end{bmatrix}$. From the eigenvalue, eigenvector relation it follows that $AP = PD$ and since P is invertible ($\det P = -7 \neq 0$) it follows that $P^{-1}AP = D$. ◀

5. [**10 Points**] Let $A = \begin{bmatrix} 5 & -12 \\ 1 & -2 \end{bmatrix}$ and $P = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}$. Then $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. (You do not need to check this.) Using this information, compute A^8.

▶ **Solution.** From the given equation it follows that

$$A = P \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} P^{-1} = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix}.$$

Then

$$\begin{aligned} A^8 = \begin{bmatrix} 5 & -12 \\ 1 & -2 \end{bmatrix}^8 &= \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^8 \begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 256 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 1024 \\ 1 & 256 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 1021 & -3060 \\ 255 & -764 \end{bmatrix}. \end{aligned}$$

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6. [**12 Points**] Determine if the following functions are linear transformations. Give reasons.

(a) $T : M_{22} \to \mathbb{R}$ defined by $T(A) = \det A$.

▶ **Solution.** This is not a linear transformation since $T(cA) \neq cT(A)$ for all $c \in \mathbb{R}$ and $A \in M_{22}$. For a specific example

$$T\left(2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) = \det \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 4 \neq 2 \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is also true that $T(A + B) \neq T(A) + T(B)$ since $\det(A + B) \neq \det A + \det B$ in general. ◀

(b) $S : P_1 \to \mathbb{R}$ defined by $S(a + bx) = a + 2b$.

▶ **Solution.** This is a linear transformation since it preserves vector addition and scalar multiplication. To see this:

$$\begin{aligned} S((a + bx) + (c + dx)) &= S((a + c) + (b + d)x) \\ &= (a + c) + 2(b + d) = (a + 2b) + (c + 2d) \\ &= S(a + bx) + S(c + dx) \end{aligned}$$

and

$$S(k(a+bx)) = S(ka+kbx) = ka+2kb = k(a+2b) = kS(a+bx).$$

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7. [**14 Points**] Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ be the basis of $\mathbb{R}^2$ with $\mathbf{v}_1 = (2, 1)$ and $\mathbf{v}_2 = (1, 1)$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation for which $T(\mathbf{v}_1) = (1, 0)$ and $T(\mathbf{v}_2) = (-1, 1)$. Find a formula for $T(x_1, x_2)$ and use it to compute $T(3, 5)$.

   ▶ **Solution.** Write the vector $\mathbf{v} = (x_1, x_2) = c_1\mathbf{v_1} + c_2\mathbf{v_2} = c_1(2, 1) + c_1(1, 1)$, which gives the system of equations

$$\begin{aligned} x_1 &= 2c_1 + c_2 \\ x_2 &= c_1 + c_2. \end{aligned}$$

   Solve for c_1 and c_2 to get $c_1 = x_1 - x_2$ and $c_2 = -x_1 + 2x_2$. Then T is a linear transformation so

$$\begin{aligned} T(x_1, x_2) &= T(c_1\mathbf{v_1} + c_2\mathbf{v_2}) \\ &= c_1T(\mathbf{v_1}) + c_2T(\mathbf{v_2}) \\ &= (x_1 - x_2)(1, 0) + (-x_1 + 2x_2)(-1, 1) \\ &= (2x_1 - 3x_2, -x_1 + 2x_2). \end{aligned}$$

   Thus, $T(3, 5) = (-9, 7)$. ◀

8. [**9 Points**] Let A be a 7×6 matrix. and assume that the dimension of the nullspace of A is 2.

   (a) What is the dimension of the column space of A?

   ▶ **Solution.** From the dimension theorem, $\dim \mathrm{Col}(A) + \dim \mathrm{Null}(A) = 6$ so $\dim \mathrm{Col}(A) = 4$. ◀

   (b) What is the rank of A^T? (Recall that A^T is the transpose of A.)

   ▶ **Solution.** $\mathrm{Rank}(A^T) = \mathrm{Rank}(A) = \dim \mathrm{Col}(A) = 4$. ◀

   (c) What is the nullity of A^T

   ▶ **Solution.** $\mathrm{Rank}(A^T) + \dim \mathrm{Null}(A^T) = 7$ =the number of columns of A^T. Thus, $\dim \mathrm{Null}(A^T) = 7 - 4 = 3$. ◀