Instructions. Answer each of the questions on your own paper, and be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper. There are 5 problems, with a total of 100 points possible. The points for each problem is listed in parentheses.

(30) 1. Consider the system of linear equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= a \\ 4x_1 + 5x_2 + 6x_3 &= b \\ 7x_1 + 8x_2 + 9x_3 &= c \end{aligned} \tag{1}$$

where a, b, and c are some real numbers.

(a) Find the general solution of the system when $a = 0$, $b = 0$, $c = 0$.

▶ **Solution.** Apply Gauss reduction to the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right] \xrightarrow[-7\rho_1+\rho_3]{-4\rho_1+\rho_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & -12 & 0 \end{array}\right]$$

$$\xrightarrow{-2\rho_2+\rho_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-\frac{1}{3}\rho_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-2\rho_2+\rho_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The last matrix is in reduced row echelon form and the and is the augmented matrix of the equivalent system:

$$\begin{aligned} x_1 \qquad - x_3 &= 0 \\ x_2 + 2x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

The leading variables are x_1 and x_2, while the remaining variable x_3 is a free variable. Solving for x_1 and x_2 in terms of x_3 gives the solution set:

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} x_3 \;\middle|\; x_3 \text{ is arbitrary.} \right\}$$

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(b) A particular solution of Equation (1) when $a = 4$, $b = 10$, $c = 16$ is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Find the general solution of Equation (1) when $a = 4$, $b = 10$, $c = 16$.

▶ **Solution.** The general solution of the nonhomogeneous equation is the sum of any particular solution and the general solution of the associated homogeneous equation. Since the general solution of the associated homogeneous equation was found in part (a), the general solution of Equation (1) when $a = 4$, $b = 10$, $c = 16$ is

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} x_3 \,\middle|\, x_3 \text{ is arbitrary.} \right\}$$

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(18) 2. In each part suppose that the augmented matrix for a system of linear equations has been reduced by row operations to the given reduced row echelon form. Solve the system

(a) $\left[\begin{array}{cccc|c} 1 & -2 & 0 & 3 & 4 \\ 0 & 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ (b) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ (c) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 8 \end{array}\right]$

▶ **Solution.** (a) This matrix is the augmented matrix of the linear system:

$$\begin{aligned} x_1 - 2x_2 \qquad + 3x_4 &= 4 \\ x_3 + 5x_4 &= -6 \end{aligned}$$

In this system the first and third variables, x_1 and x_3 are leading variables, so that the second and fourth variables x_2 and x_4, are free. Hence there are infinitely many solutions, parameterized by the free variable x_2 and x_4:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -6 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -3 \\ 0 \\ -5 \\ 1 \end{bmatrix} x_4$$

where x_2, $x_4 \in \mathbb{R}$ are arbitrary.

(b) The second row corresponds to the inconsistent equation $0 = 1$. Hence there is no solution.

(c) There is a unique solution $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 8 \end{bmatrix}$.

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(20) 3. Determine, with justification, whether each subset of $\mathbb{R}^3$ is a vector subspace.

(a) $V_1 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \,\middle|\, a = b + c \text{ and } a,\ b,\ c \in \mathbb{R} \right\}$

▶ **Solution.** V_1 is a subspace of $\mathbb{R}^3$. To see this, we check that it is closed under vector addition and scalar multiplication. Thus, let $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$ be two elements of V_1. This means that $a = b + c$ and $d = e + f$. Then

$$\vec{v} + \vec{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} a+d \\ b+e \\ c+f \end{bmatrix}.$$

Since $a + d = (b + c) + (e + f) = (b + e) + (c + f)$ it follows that $\vec{v} + \vec{w} \in V_1$. Now if r is any scalar, then $r\vec{v} = r \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} ra \\ rb \\ rc \end{bmatrix}$. Since $ra = r(b+c) = rb+rc$, it follows that $r\vec{v} \in V_1$. Therefore V_1 is closed under arbitrary vector addition and scalar multiplications. Thus V_1 is a subspace of $\mathbb{R}^3$. ◀

(b) $V_2 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \middle| \, a = b + 1 \text{ and } a,\, b,\, c \in \mathbb{R} \right\}$

▶ **Solution.** V_2 is not a subspace of $\mathbb{R}^3$ since it is not closed under vector addition (or scalar multiplication). This can be seen by means of a concrete example of $\vec{v} \in V_2$ and $\vec{w} \in V_2$, but $\vec{v} + \vec{w} \notin V_2$. Let $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. $\vec{v}$ is in V_2 since for this vector $a = 1 = 0 + 1 = b + 1$ and $\vec{w} \in V_2$ since for this vector $a = 2 = 1 + 1 = b + 1$. However, $\vec{v} + \vec{w} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \notin V_2$ since for this vector $a = 3 \neq 1 + 1 = b + 1$. ◀

(16) 4. Consider the following vectors $\vec{v_1}$, $\vec{v_2}$, $\vec{v_3}$, and $\vec{w}$ in $\mathbb{R}^3$:

$$\vec{v_1} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v_2} = \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix}, \quad \vec{v_3} = \begin{bmatrix} 3 \\ -1 \\ -3 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}.$$

(a) Determine if the vector $\vec{w}$ is in the span of the set $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$.

▶ **Solution.** It is necessary to determine if $\vec{w} = c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3}$ for some choice to scalars c_1, c_2, c_3. Thus, we need to see if we can write

$$c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}.$$

This is possible if and only if we can solve the following system of linear equations:

$$\begin{aligned} c_1 - 2c_2 + 3c_3 &= 5 \\ -c_1 - c_2 - c_3 &= -4 \\ - c_2 - 3c_3 &= -7 \end{aligned}$$

Apply Gauss-Jordan reduction to the augmented matrix of the system:

$$\left[\begin{array}{rrr|r} 1 & -2 & 3 & 5 \\ -1 & -1 & -1 & -4 \\ 0 & -1 & -3 & -7 \end{array}\right] \xrightarrow{\rho_1+\rho_2} \left[\begin{array}{rrr|r} 1 & -2 & 3 & 5 \\ 0 & -3 & 2 & 1 \\ 0 & -1 & -3 & -7 \end{array}\right]$$

$$\xrightarrow{\rho_2\leftrightarrow\rho_3} \left[\begin{array}{rrr|r} 1 & -2 & 3 & 5 \\ 0 & -1 & -3 & -7 \\ 0 & -3 & 2 & 1 \end{array}\right] \xrightarrow{-\rho_2} \left[\begin{array}{rrr|r} 1 & -2 & 3 & 5 \\ 0 & 1 & 3 & 7 \\ 0 & -3 & 2 & 1 \end{array}\right]$$

$$\xrightarrow{3\rho_2+\rho_3} \left[\begin{array}{rrr|r} 1 & -2 & 3 & 5 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 11 & 22 \end{array}\right] \xrightarrow{\frac{1}{11}\rho_3} \left[\begin{array}{rrr|r} 1 & -2 & 3 & 5 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow[-3\rho_3+\rho_1]{-3\rho_3+\rho_2} \left[\begin{array}{rrr|r} 1 & -2 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{2\rho_2+\rho_1} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

The system of equations thus has a unique solution $c_1 = 1$, $c_2 = 1$ and $c_3 = 2$ and hence $\vec{w}$ is in the span of $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$. ◀

(b) If $\vec{w}$ is in the span of the set $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$, express $\vec{w}$ as a linear combination of $\vec{v_1}$, $\vec{v_2}$, and $\vec{v_3}$. You should be able to answer this using the work done in part (a).

▶ **Solution.** From part (a), it follows that $\vec{w} = \vec{v_1} + \vec{v_2} + 2\vec{v_3}$. ◀

(16) 5. Determine if the following sets of vector are linearly independent. If **not**, write one vector as a linear combination of other vectors in the set.

(a) $\left\{ \vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 0 \\ 2 \\ -10 \end{bmatrix} \right\}$

▶ **Solution.** It is necessary to see if there is a nontrivial linear dependence relation

$$c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} = \vec{0}.$$

That is, is there a nontrivial relation

$$c_1 \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 2 \\ -10 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This is possible if and only if we can solve the following system of linear equations:

$$\begin{aligned} c_1 - c_2 &= 0 \\ 2c_1 - c_2 + 2c_3 &= 0 \\ -4c_1 - c_2 - 10c_3 &= 0 \end{aligned}$$

Apply Gauss-Jordan reduction to the augmented matrix of the system:

$$\left[\begin{array}{rrr|r} 1 & -1 & 0 & 0 \\ 2 & -1 & 2 & 0 \\ -4 & -1 & -10 & 0 \end{array}\right] \xrightarrow[4\rho_1+\rho_3]{-2\rho_1+\rho_2} \left[\begin{array}{rrr|r} 1 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -5 & -10 & 0 \end{array}\right]$$

$$\xrightarrow{5\rho_2+\rho_3} \left[\begin{array}{rrr|r} 1 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\rho_2+\rho_1} \left[\begin{array}{rrr|r} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This last matrix is the augmented matrix of the linear system

$$\begin{aligned} c_1 + 2c_3 &= 0 \\ c_2 + 2c_3 &= 0 \\ 0 &= 0 \end{aligned}$$

which has the solution set

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} c_3$$

for any value of c_3. Thus, the set is linearly dependent, and a specific linear dependence relation is

$$-2\vec{v_1} - 2\vec{v_2} + \vec{v_3} = \vec{0}.$$

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(b) $\left\{ \vec{w_1} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} -6 \\ 2 \\ -6 \end{bmatrix} \right\}$

▶ **Solution.** It is necessary to see if there is a nontrivial linear dependence relation

$$c_1\vec{w_1} + c_2\vec{v_2} + c_3\vec{v_3} = \vec{0}.$$

That is, is there a nontrivial relation

$$c_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix} + c_3 \begin{bmatrix} -6 \\ 2 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This is possible if and only if we can solve the following system of linear equations:

$$\begin{aligned} c_1 - 2c_2 - 6c_3 &= 0 \\ 2c_3 &= 0 \\ -2c_1 - 2c_2 - 6c_3 &= 0 \end{aligned}$$

The second equation already gives $c_3 = 0$. Substituting this into the other equations give a linear system

$$\begin{aligned} c_1 - 2c_2 &= 0 \\ -2c_1 - 2c_2 &= 0 \end{aligned}$$

This system is easily solved to give $c_1 = c_2 = 0$. Thus, the vectors are linearly independent.

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