Exam 1 will be on Monday, September 18, 2017. The syllabus for Exam 1 consists of Sections One.I, One.III, Two.I, and Two.II. You should know the main definitions, results and computational techniques that we have covered in these sections. The following are problems similar to those you might expect on your exam. Problems can also be similar to assigned homework problems and suggested problems from the text, so you should certainly review those. You may, and are encouraged to, bring any questions that you have to be discussed during class on Friday, September 15.

1. (a) Write the *augmented matrix* $\left[A \mid \vec{b}\right]$ for the linear system

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & a^2x_3 & = & a \\ -x_1 & & & + & x_3 & = & 3 \\ x_1 & + & x_2 & + & 9x_3 & = & -3 \end{array}$$

   (b) Use the matrix $\left[A \mid \vec{b}\right]$ A from part (a) to find all values of a for which the system has a *unique* solution.

   (c) Find all values of a for which the system has infinitely many solutions.

   (d) Find all values of a for which the system has no solutions.

2. (a) Find the *augmented matrix* $\left[A \mid \vec{b}\right]$ for the linear system

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & + & x_3 & + & x_4 & = & 1 \\ -x_1 & + & x_2 & - & x_3 & + & x_4 & = & 3 \\ 3x_1 & + & 2x_2 & + & x_3 & & & = & 7 \\ 3x_1 & - & 2x_2 & + & x_3 & & & = & -5 \end{array}$$

   (b) Solve the system in part (a) by the Gauus-Jordan method. That is, by finding rref(A).

3. In each part suppose that the augmented matrix for a system of linear equations has been reduced by row operations to the given reduced row echelon form. Solve the system

(a) $\left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$ (b) $\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 7 \end{array}\right]$ (c) $\left[\begin{array}{ccc|r} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -9 \\ 0 & 0 & 0 & 0 \end{array}\right]$

4. (a) Find all solution to the equation $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & -1 & 0 \\ 1 & 2 & 1 & 3 \end{bmatrix}, \ \vec{b} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}.$$

   (b) Does the equation $A\vec{x} = \vec{c}$ have a solution for every vector $\vec{c} \in \mathbb{R}^3$? Explain your answer.

5. (a) Let A be a 4×4 matrix and let $\vec{c} \in \mathbb{R}^4$. If the system of equations $A\vec{x} = \vec{c}$ has a *unique* solution, what can you say about the reduced echelon form matrix $\text{rref}(A)$?

   (b) Now let B be a 4×3 matrix and suppose that the system of 4 equations in 3 unknowns $B\vec{x} = \vec{c}$ has a *unique* solution. Give the reduced echelon form of the matrix B.

6. Determine, with justification, whether each subset of $\mathcal{M}_{1\times 3}$ is a vector subspace.

   (a) $V_1 = \{ \begin{bmatrix} a & b & c \end{bmatrix} \mid a + b + 2c = 0 \text{ and } a,\, b,\, c \in \mathbb{R} \}$

   (b) $V_2 = \{ \begin{bmatrix} a & b & c \end{bmatrix} \mid abc = 0 \text{ and } a,\, b,\, c \in \mathbb{R} \}$

7. Which of the following subsets of $\mathbb{R}^2$ are subspaces?

   (a) W_1 is the set of all vectors of the form $\begin{bmatrix} x \\ y \end{bmatrix}$, where $x \geq 0$ and $y \geq 0$.

   (b) W_2 is the set of all vectors of the form $x + 2y = 0$

8. Determine (with justification) if each set is linearly independent (in the natural vector space).

   (a) $\{ \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ -6 \end{pmatrix} \}$

   (b) $\{ (1\ \ 0\ \ 1), (-1\ \ 1\ \ 3), (-1\ \ 2\ \ -3) \}$

   (c) $\{ \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 4 \\ 3 & -6 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 4 \end{pmatrix} \}$

9. Determine if the given vector, is in the span of the given set, inside $\mathbb{R}^3$.

$$\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}, \quad \{ \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \}.$$

10. Determine if the following set spans $\mathcal{P}_2$, the space of quadratic polynomial functions:

$$\{ 1, , 1 + x,\ 1 + x + x^2 \}$$

11. Suppose you have a linear system of m linear equations in n variables, which in matrix form we can write as $A\vec{x} = \vec{b}$. Here A is the coefficient matrix and $\vec{b}$ is the constant part of the equations. Explain why each of the following statements is true or give a counterexample if it is false.

    (a) If $m = n$ there is at most one solution.

    (b) If $n > m$ you can always solve $A\vec{x} = \vec{b}$.

    (c) If $n > m$ the associated homogeneous equation $A\vec{x} = \vec{0}$ has an infinite number of solutions.

    (d) If $n < m$ the only solution of $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$.

12. Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.

    (a) Find the general solution of the homogeneous equation $A\vec{x} = \vec{0}$.

    (b) Find some solution of $A\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

    (c) Find the general solution of the equation in part (b).

    (d) Find some solution of $A\vec{x} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

    (e) Find some solution of $A\vec{x} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$. [Note: $\begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 0 \end{bmatrix}$]

    [Remark: After you have done parts (a), (b), and (d), it is possible to immediately write down the solutions to the remaining parts.]