Exam 1 will be on Monday, September 18, 2017. The syllabus for Exam 1 consists of Sections One.I, One.III, Two.I, and Two.II. You should know the main definitions, results and computational techniques that we have covered in these sections. The following are problems similar to those you might expect on your exam. Problems can also be similar to assigned homework problems and suggested problems from the text, so you should certainly review those. You may, and are encouraged to, bring any questions that you have to be discussed during class on Friday, September 15.

1. (a) Write the *augmented matrix* $\left[A \mid \vec{b}\right]$ for the linear system

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & a^2x_3 & = & a \\ -x_1 & & & + & x_3 & = & 3 \\ x_1 & + & x_2 & + & 9x_3 & = & -3 \end{array}$$

▶ **Solution.**

$$\left[A \mid \vec{b}\right] = \left[\begin{array}{rrr|r} 1 & 1 & a^2 & a \\ -1 & 0 & 1 & 3 \\ 1 & 1 & 9 & -3 \end{array}\right]$$

◀

(b) Use the matrix $\left[A \mid \vec{b}\right]$ from part (a) to find all values of a for which the system has a *unique* solution.

▶ **Solution.** Apply Gauss reduction to the augmented matrix $\left[A \mid \vec{b}\right]$:

$$\left[\begin{array}{rrr|r} 1 & 1 & a^2 & a \\ -1 & 0 & 1 & 3 \\ 1 & 1 & 9 & -3 \end{array}\right] \xrightarrow[-\rho_1+\rho_3]{\rho_1+\rho_2} \left[\begin{array}{ccc|c} 1 & 1 & a^2 & a \\ 0 & 1 & a^2+1 & a+3 \\ 0 & 0 & 9-a^2 & -3-a \end{array}\right]$$

If $9 - a^2 \neq 0$, then the last matrix is in echelon form and all variables are leading variables, so there is a unique solution in this case. Thus, there is a unique solution for all choices of $a \neq \pm 3$. ◀

(c) Find all values of a for which the system has infinitely many solutions.

▶ **Solution.** The system will have infinitely many solutions, provided the last row of the last matrix is $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \end{array}\right]$. This will happen when both $9 - a^2 = 0$ and $-3 - a = 0$, which happens only if $a = -3$. ◀

(d) Find all values of a for which the system has no solutions.

▶ **Solution.** If $a = 3$ the last row of the last matrix is $\left[\begin{array}{ccc|c} 0 & 0 & 0 & -6 \end{array}\right]$. This represents an inconsistent equation $0 = -6$, and hence there is no solution in case $a = 3$. ◀

2. (a) Find the *augmented matrix* $\left[A \mid \vec{b}\right]$ for the linear system

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & + & x_3 & + & x_4 & = & 1 \\ -x_1 & + & x_2 & - & x_3 & + & x_4 & = & 3 \\ 3x_1 & + & 2x_2 & + & x_3 & & & = & 7 \\ 3x_1 & - & 2x_2 & + & x_3 & & & = & -5 \end{array}$$

▶ **Solution.**

$$\left[A \mid \vec{b}\right] = \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 3 \\ 3 & 2 & 1 & 0 & 7 \\ 3 & -2 & 1 & 0 & -5 \end{array}\right]$$

◀

(b) Solve the system in part (a) by the Gauss-Jordan method. That is, by finding rref(A).

▶ **Solution.** Apply Gauss reduction to the augmented matrix $\left[A \mid \vec{b}\right]$:

$$\left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 3 \\ 3 & 2 & 1 & 0 & 7 \\ 3 & -2 & 1 & 0 & -5 \end{array}\right] \xrightarrow[\substack{-3\rho_1+\rho_3 \\ -3\rho_1+\rho_4}]{\rho_1+\rho_2} \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 2 & 4 \\ 0 & -1 & -2 & -3 & 4 \\ 0 & -5 & -2 & -3 & -8 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}\rho_2} \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & -1 & -2 & -3 & 4 \\ 0 & -5 & -2 & -3 & -8 \end{array}\right] \xrightarrow[5\rho_2+\rho_4]{\rho_2+\rho_3} \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & -2 & -2 & 6 \\ 0 & 0 & -2 & 2 & 2 \end{array}\right]$$

$$\xrightarrow[-\frac{1}{2}\rho_4]{-\frac{1}{2}\rho_3} \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & -1 & -1 \end{array}\right] \xrightarrow[-\rho_3+\rho_4]{} \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & -2 & 2 \end{array}\right]$$

$$\xrightarrow[-\frac{1}{2}\rho_4]{} \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow[\substack{-\rho_4+\rho_2 \\ -\rho_4+\rho_1}]{-\rho_4+\rho_3} \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow[-\rho_3+\rho_1]{} \left[\begin{array}{rrrr|r} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow[-\rho_2+\rho_1]{} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

The last matrix is in reduced row echelon form and the solution can be read off directly:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \\ -1 \end{bmatrix}$$

◀

3. In each part suppose that the augmented matrix for a system of linear equations has been reduced by row operations to the given reduced row echelon form. Solve the system

(a) $\left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$ (b) $\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 7 \end{array}\right]$ (c) $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -9 \\ 0 & 0 & 0 & 0 \end{array}\right]$

▶ **Solution.** (a) The last row corresponds to the inconsistent equation $0 = 1$. Hence there is not solution.

(b) There is a unique solution $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$

(c) In this case the first and third variables are leading variables, so that the second variable is free. Hence there are infinitely many solutions, parameterized by the free variable y:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -9 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} y$$

where $y \in \mathbb{R}$ is arbitrary.

◀

4. (a) Find all solution to the equation $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & -1 & 0 \\ 1 & 2 & 1 & 3 \end{bmatrix}, \ \vec{b} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}.$$

▶ **Solution.** Apply Gauss reduction to the augmented matrix $\left[A \mid \vec{b}\right]$:

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 2 & 4 & -1 & 0 & -1 \\ 1 & 2 & 1 & 3 & 4 \end{array}\right] \xrightarrow[-\rho_1+\rho_3]{-2\rho_1+\rho_2} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right]$$

$$\xrightarrow{-\rho_2} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right] \xrightarrow{-\rho_2+\rho_3} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The last matrix is in reduced row echelon form and is the augmented matrix of the following linear system.

$$\begin{aligned} x_1 + 2x_2 \quad + \ x_4 &= 1 \\ x_3 + 2x_4 &= 3 \\ &= 0 \end{aligned}$$

In this system the leading variables are x_1 nd x_3, with the remaining two variables x_2 and x_4 being free variables. The solutions can be read off directly and parameterized by the free variables x_2 and x_4 as follows:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix} x_4.$$

The parameters x_2 and x_4 are arbitrary real numbers. ◀

(b) Does the equation $A\vec{x} = \vec{c}$ have a solution for every vector $\vec{c} \in \mathbb{R}^3$? Explain your answer.

▶ **Solution.** The equation does not have a solution for every vector $\vec{c} \in \mathbb{R}^3$. The reason is that if you try to replace the vector $\vec{b} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$ with an arbitrary vector $\vec{c} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$, the Gauss-Jordan reduction will still lead to the same matrix corresponding to the coefficient part of the equation, so that the third row of the reduced augmented matrix will have the form $\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & d \end{array}\right]$ but the last entry d will not be 0 unless the elements α, β, and γ have a particular relationship. In this case, tracing through the row operations will give $d = \beta + \gamma - 3\alpha$, and for most $\vec{c}$, the resulting $d \neq 0$, which means $A\vec{x} = \vec{c}$ is not solvable. ◀

5. (a) Let A be a 4×4 matrix and let $\vec{c} \in \mathbb{R}^4$. If the system of equations $A\vec{x} = \vec{c}$ has a *unique* solution, what can you say about the reduced echelon form matrix rref(A)?

▶ **Solution.** If there is a unique solution then each of the 4 rows of rref(A) must have a leading 1, and since there are only 4 columns, each column contains a leading 1, with the other entries in the column being 0. Thus

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

◀

(b) Now let B be a 4×3 matrix and suppose that the system of 4 equations in 3 unknowns $B\vec{x} = \vec{c}$ has a *unique* solution. Give the reduced echelon form of the matrix B.

▶ **Solution.** If B is 4×3, then $B\vec{x} = \vec{c}$ is a system of 4 equations in 3 unknowns. If there is a unique solution, then there must be a solution so no rows of form $\left[\begin{array}{ccc|c} 0 & 0 & 0 & s \end{array}\right]$ with $s \neq 0$ can be in the reduced echelon form of the augmented matrix $\begin{bmatrix} B \mid \vec{c} \end{bmatrix}$. Moreover, there can be no free variables, so each of the three columns corresponding to a variable must have a leading 1 and all other entries in the column are 0. Thus

$$\operatorname{rref}(\begin{bmatrix} B \mid \vec{c} \end{bmatrix}) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & a_2 \\ 0 & 0 & 1 & a_3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The first three columns is the reduced echelon form of B. ◀

6. Determine, with justification, whether each subset of $\mathcal{M}_{1\times 3}$ is a vector subspace.

   (a) $V_1 = \{ \begin{bmatrix} a & b & c \end{bmatrix} \mid a + b + 2c = 0 \text{ and } a,\, b,\, c \in \mathbb{R} \}$

   ▶ **Solution.** V_1 is a subspace. To see this, we check that it is closed under vector addition and scalar multiplication. Thus, let $\vec{v} = \begin{bmatrix} a & b & c \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} d & e & f \end{bmatrix}$ be two elements of V_1. This means that $a + b + 2c = 0$ and $d + e + 2f = 0$. Then

   $$\vec{v} + \vec{w} = \begin{bmatrix} a & b & c \end{bmatrix} + \begin{bmatrix} d & e & f \end{bmatrix} = \begin{bmatrix} a+d & b+e & c+f \end{bmatrix}.$$

   Since $(a+d)+(b+e)+2(c+f) = (a+b+2c)+(d+e+2f) = 0+0 = 0$ it follows that $\vec{v} + \vec{w} \in V_1$. Now if r is any scalar, then $r\vec{v} = r\begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} ra & rb & rc \end{bmatrix}$. Since $(ra) + (rb) + 2(rc) = r(a + b + 2c) = r \cdot 0 = 0$, it follows that $r\vec{v} \in V_1$. Therefore V_1 is closed under arbitrary vector addition and scalar multiplications. Thus V_1 is a subspace of $\mathcal{M}_{1\times 3}$. ◀

   (b) $V_2 = \{ \begin{bmatrix} a & b & c \end{bmatrix} \mid abc = 0 \text{ and } a,\, b,\, c \in \mathbb{R} \}$

   ▶ **Solution.** V_2 is not a subspace since it is not closed under vector addition. To see this, let $\vec{v} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ and let $\vec{w} = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$. Both $\vec{v}$ and $\vec{w}$ are in V_2, since the product of the three elements in each is 0. However,

   $$\vec{v} + \vec{w} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix},$$

   and the product of the three elements of $\vec{v} + \vec{w}$ is $1 \neq 0$, so the sum is not in V_2. Therefore, V_2 is not closed under vector addition, so it is not a subspace. ◀

7. Which of the following subsets of $\mathbb{R}^2$ are subspaces?

   (a) W_1 is the set of all vectors of the form $\begin{bmatrix} x \\ y \end{bmatrix}$, where $x \geq 0$ and $y \geq 0$.

   ▶ **Solution.** Let $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $\vec{v} \in W_1$ since both entries are ≥ 0. However, $(-1)\vec{v} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ is not in W_1 since the first entry is < 0. Thus, W_1 is not closed under scalar multiplication, so W_1 is not a subspace of $\mathbb{R}^2$. ◀

(b) W_2 is the set of all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ with $x + 2y = 0$.

▶ **Solution.** W_2 is a subspace of $\mathbb{R}^2$. The argument is essentially identical to that for Exercise 6 (a). ◀

8. Determine (with justification) if each set is linearly independent (in the natural vector space).

(a) $\{ \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ -6 \end{pmatrix} \}$

▶ **Solution.** This set is linearly dependent, since the second vector is -2 times the first vector. ◀

(b) $\{ (1 \;\; 0 \;\; 1), (-1 \;\; 1 \;\; 3), (-1 \;\; 2 \;\; -3) \}$

▶ **Solution.** Suppose there is a linear dependence relation

$$c_1(1 \;\; 0 \;\; 1) + c_2(-1 \;\; 1 \;\; 3) + c_3(-1 \;\; 2 \;\; -3) = (0 \;\; 0 \;\; 0).$$

Adding the left hand side together gives a system of linear equations for c_1, c_2, c_3:

$$\begin{aligned} c_1 - \;\; c_2 - \;\; c_3 &= 0 \\ c_2 + 2c_3 &= 0 \\ c_1 + 3c_2 - 3c_3 &= 0 \end{aligned}$$

The augmented matrix of this system is $\left[\begin{array}{rrr|r} 1 & -1 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 3 & -3 & 0 \end{array}\right]$, which after Gauss reduction, produces the echelon matrix $\left[\begin{array}{rrr|r} 1 & -1 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right]$. This gives the equivalent system of linear equations

$$\begin{aligned} c_1 - c_2 - \;\; c_3 &= 0 \\ c_2 + 2c_3 &= 0 \\ -\, 6c_3 &= 0 \end{aligned}$$

The only solution of this system of equations is $c_1 = c_2 = c_3 = 0$, so the set of vectors is linearly independent. ◀

(c) $\{ \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 4 \\ 3 & -6 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 4 \end{pmatrix} \}$

▶ **Solution.** Suppose there is a linear dependence relation

$$c_1 \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix} + c_2 \begin{pmatrix} 3 & 4 \\ 3 & -6 \end{pmatrix} + c_3 \begin{pmatrix} 1 & 0 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Adding the left hand side together gives a system of linear equations for c_1, c_2, c_3:

$$\begin{aligned} 5c_1 + 3c_2 + \ \ c_3 &= 0 \\ 4c_1 + \ \ c_2 \qquad &= 0 \\ c_1 + 3c_2 - 3c_3 &= 0 \\ 2c_1 - 6c_2 + 4c_3 &= 0 \end{aligned}$$

The augmented matrix of this system is $\left[\begin{array}{ccc|c} 5 & 3 & 1 & 0 \\ 4 & 4 & 0 & 0 \\ 1 & 3 & -1 & 0 \\ 2 & -6 & 4 & 0 \end{array}\right]$, which after Gauss-Jordan reduction, produces the reduced echelon matrix $\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. This gives the equivalent system of linear equations

$$\begin{aligned} c_1 \qquad + \tfrac{1}{2}c_3 &= 0 \\ c_2 - \tfrac{1}{2}c_3 &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

c_3 is a free variable, so there are infinitely many solutions. Letting $c_3 = 2$ gives one solution $c_1 = -1$, $c_2 = 1$, $c_3 = 2$. Thus, there is a nontrivial linear dependence relation

$$(-1)\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix} + (1)\begin{pmatrix} 3 & 4 \\ 3 & -6 \end{pmatrix} + (2)\begin{pmatrix} 1 & 0 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

so the set is not linearly independent. ◀

9. Determine if the given vector, is in the span of the given set, inside $\mathbb{R}^3$.

$$\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}, \quad \{\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\}.$$

▶ **Solution.** Try to write:

$$c_1\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}.$$

If this is possible, then c_1 and c_2 must satisfy the system of linear equations

$$\begin{aligned} 2c_1 + c_2 &= 3 \\ c_1 - c_2 &= 0. \\ -c_1 + c_2 &= 3 \end{aligned}$$

The augmented matrix of this system is

$$\left[\begin{array}{rr|r} 2 & 1 & 3 \\ 1 & -1 & 2 \\ -1 & 1 & -2 \end{array}\right].$$

Gauss-Jordan reduction gives the reduced echelon form of the augmented matrix:

$$\begin{bmatrix} 1 & 0 & \frac{5}{3} \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}.$$

This gives $c_1 = \frac{5}{3}$ and $c_2 = -\frac{1}{3}$ so that

$$\frac{5}{3}\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}.$$

Therefore, the first vector is in the span of the given set.

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10. Determine if the following set spans $\mathcal{P}_2$, the space of quadratic polynomial functions:

$$\{1,\ 1+x,\ 1+x+x^2\}$$

▶ **Solution.** Let $p(x) = a + bx + cx^2$ be an arbitrary element of $\mathcal{P}_2$. Try to write

$$p(x) = c_1 \cdot 1 + c_2(1+x) + c_3(1+x+x^2).$$

Comparing coefficients of corresponding powers of x on the left and the right gives a linear system of equations for c_1, c_2, c_3:

$$\begin{aligned} c_1 + c_2 + c_3 &= a \\ c_2 + c_3 &= b \\ c_3 &= c \end{aligned}$$

This system is already in echelon form, so it can be solved by back substitution to get $c_1 = a - b$, $c_2 = b - c$, $c_3 = c$. Therefore, we can write

$$p(x) = a + bx + c^2 = (a-b)\cdot 1 + (b-c)(1+x) + c(1+x+x^2).$$

Since $p(x)$ is arbitrary, this means that the given set spans $\mathcal{P}_2$. ◀

11. Suppose you have a linear system of m linear equations in n variables, which in matrix form we can write as $A\vec{x} = \vec{b}$. Here A is the coefficient matrix and $\vec{b}$ is the constant part of the equations. Explain why each of the following statements is true or give a counterexample if it is false.

(a) If $m = n$ there is at most one solution.

▶ **Solution. False.** Counterexample: $A\vec{x} = \vec{0}$ for $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ has an infinite number of solutions, namely $\begin{bmatrix} 1 \\ -1 \end{bmatrix} r$ for all real numbers r. ◀

(b) If $n > m$ you can always solve $A\vec{x} = \vec{b}$.

▶ **Solution. False.** Counterexample: The system of 2 equations in 1 unknown

$$\begin{aligned} x &= 0 \\ x &= 1 \end{aligned}$$

has no solution. ◀

(c) If $n > m$ the associated homogeneous equation $A\vec{x} = \vec{0}$ has an infinite number of solutions.

▶ **Solution. True.** A homogeneous system always has at least one solution, namely $\vec{x} = \vec{0}$. If the number of variables n is greater that the number of equations, then when the system is put in reduced echelon form, not every variable can be the leading variable of an equation, so there will have to be some free variables. Assigning each free variable an arbitrary value and solving for the leading variables with these given values produces infinitely many solutions to the homogeneous system. ◀

(d) If $n < m$ the only solution of $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$.

▶ **Solution. False.** If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$, the system $A\vec{x} = \vec{0}$ has an infinite number of solutions, namely $\begin{bmatrix} 1 \\ -1 \end{bmatrix} r$ for all real numbers r. ◀

12. Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.

(a) Find the general solution of the homogeneous equation $A\vec{x} = \vec{0}$.

▶ **Solution.** Use Gauss-Jordan reduction to calculate that the reduced echelon form of A is

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{3}{2} \end{bmatrix}.$$

This is the coefficient matrix of the homogeneous system

$$\begin{aligned} x + \quad - 2z &= 0 \\ y + \quad z &= 0 \end{aligned}.$$

Hence, the solution of the original system is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \\ 1 \end{bmatrix} z$$

where z is arbitrary. ◀

(b) Find some solution of $A\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

▶ **Solution.** By inspection, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a solution of $A\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. ◀

(c) Find the general solution of the equation in part (b).

▶ **Solution.** The general solution is the sum of a particular solution and the general solution of the associated homogeneous equation. Thus

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \\ 1 \end{bmatrix} z$$

where z is arbitrary is the general solution. ◀

(d) Find some solution of $A\vec{x} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

▶ **Solution.** By inspection (or apply Gauss-Jordan reduction), $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ is a solution of $A\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. ◀

(e) Find some solution of $A\vec{x} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$. [Note: $\begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 0 \end{bmatrix}$]

▶ **Solution.** Since $\begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ a solution for $A\vec{x} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$ is obtained as a solution to $A\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ plus 2 times a solution to $A\vec{x} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$. That is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 3 \end{bmatrix}$$

is a solution to $A\vec{x} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$. ◀

[Remark: After you have done parts (a), (b), and (d), it is possible to immediately write down the solutions to the remaining parts.]