**Instructions.** Answer each of the questions on your own paper, and be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper. There are 5 problems, with a total of 70 points possible. The points for each problem is listed in parentheses.

**(20)** 1. Let $A = \begin{bmatrix} -1 & -2 & 2 & 5 & 3 \\ 1 & 2 & 0 & 3 & 1 \\ 2 & 4 & 1 & 10 & 4 \end{bmatrix}$ and let $R = \begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. You may assume that $R$ is the reduced row-echelon form of $A$, that is, $R = \text{rref}(A)$. Using this fact, answer the following questions.

(a) Find a basis of the column space of $A$.

(b) Find a basis of the row space of $A$.

(c) Find a basis of the null space of $A$.

(d) What is the rank of $A$?

(e) What is the nullity of $A$?

**(12)** 2. The list of 5 vectors $\mathcal{B} = \langle \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5 \rangle$ with

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \ \vec{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \ \vec{v}_4 = \begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix}, \ \vec{v}_5 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

spans $\mathbb{R}^3$. Find a subset of $\mathcal{B}$ that is a basis of $\mathbb{R}^3$.

**(12)** 3.

(a) Fill in the blanks in the following statement to give the statement of the **Rank-nullity theorem**:

*For any homomorphism $h : V \to W$ between vector spaces $V$ and $W$, the equation*

$$\dim \boxed{\phantom{xxxxxxxx}} + \dim \boxed{\phantom{xxxxxxxx}} = \dim \boxed{\phantom{xxxxxxxx}}$$

*holds.*

(b) Let $h : \mathbb{R}^4 \to \mathbb{R}^3$ be a homomorphism. What are the possible values of the dimension of the range of $h$.?

(c) Let $h : \mathbb{R}^4 \to \mathbb{R}^3$ be a homomorphism. What are the possible values of $\dim(\text{Ker}\, h)$?

**(16)** 4. Define a homomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$h\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 + 2x_3 \\ -x_1 - 2x_2 + x_3 \\ 2x_1 + x_2 + x_3 \end{bmatrix}.$$

(a) Is $h$ one-to-one? Explain.

(b) Is $h$ onto? Explain.

(c) If possible, find a vector $\vec{v}$ such that $h(\vec{v}) = \vec{b} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$.

**(10)** 5. **True/False.** For each sentence, write the whole word "True" if the sentence is a true statement, or the word "False" if the sentence is false. Justifications are not required for this problem. For this problem only, the answer can be written on the exam paper.

________(a) If $h : V \to W$ is a homomorphism, and if the vectors $h(\vec{v}_1), \ldots, h(\vec{v}_k)$ are linearly independent in $W$, then the vectors $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent in $V$.

________(b) If $h : V \to W$ is a homomorphism and $\dim V = \dim W$, then $h$ is an isomorphism.

________(c) If $h : \mathbb{R}^n \to \mathbb{R}^m$ is a homomorphism with $\text{rank}(h) = m$, then $h$ is an onto map. That is, the range of $h$ is $\mathbb{R}^m$.

________(d) The vector space $\mathcal{M}_{2\times 6}$ of $2 \times 6$ matrices is isomorphic to the vector space $\mathcal{M}_{3\times 4}$ of $3 \times 4$ matrices.

________(e) If $\mathcal{B}$ is a list of vectors in a vector space $V$ such that $\mathcal{B}$ spans $V$, then every vector in $V$ can be written as a linear combination of vectors from $\mathcal{B}$ in only one way.