Instructions. Answer each of the questions on your own paper, and be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper. There are 5 problems, with a total of 70 points possible. The points for each problem is listed in parentheses.

(20) 1. Let $A = \begin{bmatrix} -1 & -2 & 2 & 5 & 3 \\ 1 & 2 & 0 & 3 & 1 \\ 2 & 4 & 1 & 10 & 4 \end{bmatrix}$ and let $R = \begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. You may assume that R is the reduced row-echelon form of A, that is, $R = $rref$(A)$. Using this fact, answer the following questions.

(a) Find a basis of the column space of A.

(b) Find a basis of the row space of A.

(c) Find a basis of the null space of A.

(d) What is the rank of A?

(e) What is the nullity of A?

▶ **Solution.** (a) The leading 1's of R appear in columns 1 and 3 so the corresponding columns of A form a basis of the column space of A:

$$\mathcal{B}_{\mathrm{Col}(A)} = \left\langle \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\rangle.$$

(b) The nonzero rows of R form a basis of the row space of A:

$$\mathcal{B}_{\mathrm{Row}(A)} = \left\langle \begin{bmatrix} 1 & 2 & 0 & 3 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 4 & 2 \end{bmatrix} \right\rangle.$$

(c) The null space of A is the same as the null space of R and the homogeneous system $R\vec{x} = \vec{0}$ is

$$\begin{aligned} x_1 + 2x_2 \quad + 3x_3 + \quad x_5 &= 0 \\ x_3 + 4x_4 + 2x_5 &= 0 \end{aligned}$$

The free variables for this system are x_2, x_4, and x_5, so the null space of A, which is the solution set of this system is

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 3x_4 - x_5 \\ x_2 \\ -4x_4 - 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} -1 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} x_5 \right\}$$

where x_2, x_4, and x_5 are arbitrary. Thus a basis of the null space is

$$\mathcal{B}_{\mathrm{Null}(A)} = \left\langle \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\rangle$$

(d) The rank of A is the dimension of the row space of A, which is equal to the dimension of the column space of A. Thus the rank of A is 2 from part (a) or part (b).

(e) The nullity of A is the dimension of the null space of A, which is 3 from part (c). ◀

(12) 2. The list of 5 vectors $\mathcal{B} = \langle \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5 \rangle$ with

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \ \vec{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \ \vec{v}_4 = \begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix}, \ \vec{v}_5 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

spans $\mathbb{R}^3$. Find a subset of $\mathcal{B}$ that is a basis of $\mathbb{R}^3$.

▶ **Solution.** One method is to form a matrix whose columns are the given vectors and then find a basis of the column space. This produces a basis which is a subset of the original columns. Thus, form the matrix $A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 \end{bmatrix}$ and row reduce it to find a basis of the column space.

$$\begin{bmatrix} 1 & 1 & 3 & 0 & 2 \\ -3 & 2 & 1 & 5 & -1 \\ -2 & 2 & 2 & 4 & 1 \end{bmatrix} \xrightarrow[2\rho_1+\rho_3]{3\rho_1+\rho_2} \begin{bmatrix} 1 & 1 & 3 & 0 & 2 \\ 0 & 5 & 10 & 5 & 5 \\ 0 & 4 & 8 & 4 & 5 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{5}\rho_2} \begin{bmatrix} 1 & 1 & 3 & 0 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 4 & 8 & 4 & 5 \end{bmatrix} \xrightarrow{-4\rho_2+\rho_3} \begin{bmatrix} 1 & 1 & 3 & 0 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[-2\rho_3+\rho_1]{-\rho_3+\rho_2} \begin{bmatrix} 1 & 1 & 3 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-\rho_2+\rho_1} \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The last matrix is in reduced row echelon form with leading 1's in columns 1, 2, and 5 so the corresponding columns of A form a basis for the column space of A, which is $\mathbb{R}^3$. Thus, a basis of $\mathbb{R}^3$ from among the given vectors is $\langle \vec{v}_1, \vec{v}_2, \vec{v}_5 \rangle$.

Alternatively, one can look for any vectors that are linear combinations of the preceding vectors and delete them from the list. It is clear by inspection that $\vec{v}_2$ is not in the span of $\vec{v}_1$ since it is not a scalar multiple of $\vec{v}_1$. However, $\vec{v}_3 = \vec{v}_1 + 2\vec{v}_2$ and $\vec{v}_4 = \vec{v}_2 - \vec{v}_1$ so $\vec{v}_3$ and $\vec{v}_4$ can be eliminated without changing the span of $\mathcal{B}$. Since the span of $\mathcal{B}$ is $\mathbb{R}^3$, we again arrive at a basis of $\mathbb{R}^3$ from among the given vectors is $\langle \vec{v}_1, \vec{v}_2, \vec{v}_5 \rangle$. ◀

(12) 3.

(a) Fill in the blanks in the following statement to give the statement of the **Rank-nullity theorem**:

For any homomorphism $h : V \to W$ between vector spaces V and W, the equation

$$\dim \boxed{\textbf{Range } (h)} + \dim \boxed{\textbf{Ker } (h)} = \dim \boxed{\textbf{Domain } (h)}$$

holds.

(b) Let $h : \mathbb{R}^4 \to \mathbb{R}^3$ be a homomorphism. What are the possible values of the dimension of the range of h.?

(c) Let $h : \mathbb{R}^4 \to \mathbb{R}^3$ be a homomorphism. What are the possible values of $\dim(\operatorname{Ker} h)$?

▶ **Solution.** (b) Since the range of h is a subspace of $\mathbb{R}^3$, the dimension must be less than or equal to the dimension of $\mathbb{R}^3$, which is 3. Since the dimension of the domain is at least 3, all dimensions ≤ 3 are possible. Thus, the dimension of the range of h is 0, 1, 2, or 3. For concrete examples of homomorphisms h with each of these dimensions for the range:

$$h_0(\vec{x}) = \vec{0}, \quad h_1(\vec{x}) = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix}, \quad h_2(\vec{x}) = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}, \quad h_3(\vec{x}) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(c) From the rank-nullity theorem, $\dim(\text{range}(h)) + \dim(\ker(h)) = \dim(\text{domain}(h)) = 4$. Since $0 \leq \dim(\text{range}(h)) \leq 3$, it follows that $1 \leq \dim(\ker(h)) \leq 4$. and all of these are possible. See the examples in part (b) for examples.

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(16) 4. Define a homomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$h\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 + 2x_3 \\ -x_1 - 2x_2 + x_3 \\ 2x_1 + x_2 + x_3 \end{bmatrix}.$$

(a) Is h one-to-one? Explain.

(b) Is h onto? Explain.

(c) If possible, find a vector $\vec{v}$ such that $h(\vec{v}) = \vec{b} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$.

▶ **Solution.** (a) The homomorphism h is one-to-one if and only if $\ker(h) = \left\{\vec{0}\right\}$. But $\vec{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \ker(h)$ if and only if x_1, x_2, x_3 are a solution to the homogeneous linear system

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 0 \\ -x_1 - 2x_2 + x_3 &= 0 \\ 2x_1 + x_2 + x_3 &= 0 \end{aligned}$$

Solve this by row reducing the coefficient matrix A:

$$\begin{bmatrix} 1 & -1 & 2 \\ -1 & -2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \xrightarrow[-2\rho_1+\rho_3]{\rho_1+\rho_2} \begin{bmatrix} 1 & -1 & 2 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix}$$
$$\xrightarrow[\rho_2+\rho_3]{-\frac{1}{3}\rho_2} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\rho_2+\rho_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

This last matrix has rank 2, so $\dim(\ker(h)) = 1$. This means $\ker(h) \neq \left\{\vec{0}\right\}$ so that h is not one-to-one. In particular $h\left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(b) Since $\dim(\text{Range}(h)) + \dim(\ker(h)) = \dim(\text{Domain}(h)) = 3$ and $\dim(\ker(h)) = 1$ by part (a), it follows that $\dim(\text{Range}(h)) = 2$ so $\text{Range}(h) \neq \mathbb{R}^3$. Hence h is not onto.

(c) There is a vector $\vec{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ such that $h(\vec{v}) = \vec{b} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ if and only if x_1, x_2, x_3 are a solution to the linear system

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 2 \\ -x_1 - 2x_2 + x_3 &= -1 \\ 2x_1 + x_2 + x_3 &= 3 \end{aligned}$$

Solve this by row reducing the augmented matrix $\left[A \mid \vec{b}\right]$:

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ -1 & -2 & 1 & -1 \\ 2 & 1 & 1 & 3 \end{array}\right] \xrightarrow[-2\rho_1+\rho_3]{\rho_1+\rho_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & -3 & 3 & 1 \\ 0 & 3 & -3 & -1 \end{array}\right]$$
$$\xrightarrow[\rho_2+\rho_3]{-\frac{1}{3}\rho_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 1 & -1 & -\frac{1}{3} \\ 0 & 0 & 0 & \end{array}\right] \xrightarrow{\rho_2+\rho_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & \frac{5}{3} \\ 0 & 1 & -1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This system is solvable so $\vec{b}$ exists. All solutions can be found from the last reduced row echelon matrix by using the free variable x_3:

$$\vec{b} = \begin{bmatrix} \frac{5}{3} - x_3 \\ -\frac{1}{3} + x_3 \\ x_3 \end{bmatrix}$$

where x_3 is arbitrary. For a specific $\vec{b}$ take $x_3 = 0$ for example, which gives $\vec{b} = \begin{bmatrix} \frac{5}{3} \\ -\frac{1}{3} \\ 0 \end{bmatrix}$.

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(10) 5. **True/False.** For each sentence, write the whole word "True" if the sentence is a true statement, or the word "False" if the sentence is false. Justifications are not required for this problem. For this problem only, the answer can be written on the exam paper.

True **(a)** If $h : V \to W$ is a homomorphism, and if the vectors $h(\vec{v}_1), \ldots, h(\vec{v}_k)$ are linearly independent in W, then the vectors $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent in V.

False **(b)** If $h : V \to W$ is a homomorphism and $\dim V = \dim W$, then h is an isomorphism.

True **(c)** If $h : \mathbb{R}^n \to \mathbb{R}^m$ is a homomorphism with $\mathrm{rank}(h) = m$, then h is an onto map. That is, the range of h is $\mathbb{R}^m$.

True **(d)** The vector space $\mathcal{M}_{2\times 6}$ of 2×6 matrices is isomorphic to the vector space $\mathcal{M}_{3\times 4}$ of 3×4 matrices.

False **(e)** If $\mathcal{B}$ is a list of vectors in a vector space V such that $\mathcal{B}$ spans V, then every vector in V can be written as a linear combination of vectors from $\mathcal{B}$ in only one way.