Exam 1 will be on Monday, October 16, 2017. The syllabus for Exam 2 consists of Sections Two.III.1, Two.III.2, Two.III.3, Three.I, and Three.II. You should know the main definitions, results and computational techniques that we have covered in these sections. Of The following are problems similar to those you might expect on your exam. Problems can also be similar to assigned homework problems and suggested problems from the text, so you should certainly review those. You may, and are encouraged to, bring any questions that you have to be discussed during class on Friday, October 13.

1. Let $A = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 6 & 9 & 6 & 3 \\ 1 & 2 & 4 & 1 & 2 \\ 2 & 4 & 9 & 1 & 2 \end{bmatrix}$ and let $R = \begin{bmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. You may assume that $R$ is the reduced row-echelon form of $A$, that is, $R = \mathrm{rref}(A)$. Using this fact, answer the following questions.
   (a) Find a basis of the column space of $A$.
   (b) Find a basis of the row space of $A$.
   (c) Find a basis of the null space of $A$.
   (d) What is the rank of $A$?

2. Which of the following sets of vectors are bases for $\mathbb{R}^2$?
   (a) $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$
   (b) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$
   (c) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}$
   (d) $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$
   (e) $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$
   (f) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

3. A square matrix $A$ is *symmetric* if for all indices $i$ and $j$, the entry $a_{i,j}$ equals the entry $a_{j,i}$. The symmetric $n \times n$ matrices are denoted $\mathrm{Sym}_n$. $\mathrm{Sym}_n$ is a subspace of $\mathcal{M}_{n\times n}$.
   (a) Find a basis of $\mathrm{Sym}_2$. What is the dimension?
   (b) Find a basis of $\mathrm{Sym}_3$. What is the dimension?
   (c) Find a basis of $\mathrm{Sym}_n$. What is the dimension?

4. Let $V = \mathrm{Span}(\mathcal{B})$ where $\mathcal{B} = \langle \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \rangle$ with
$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix}, \ \vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \ \vec{v}_4 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}.$$
   Find a subset of $\mathcal{B}$ that is a basis of $V$. What is the dimension of $V$?

5. Find a basis for the intersection of the two planes through the origin in $\mathbb{R}^3$:

$$x - y + 2z = 0, \quad x + 2y - z = 0.$$

6. Suppose the null space of a $5 \times 6$ matrix $A$ is 4-dimensional.
   (a) What is the dimension of the row space of $A$?
   (b) For what value of $k$ is the column space of $A$ a subspace of $\mathbb{R}^k$?
   (c) For what value of $k$ is the null space of $A$ s subspace of $\mathbb{R}^k$?

7. Consider the function $f : \mathbb{R}^3 \to \mathbb{R}^4$ given by

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 + x_3 \\ -3x_1 + 6x_2 + x_3 \\ -x_1 + 2x_2 \\ 2x_1 - 4x_2 + x_3 \end{bmatrix}.$$

   (a) Find a basis for the range of $f$. What is the rank of $f$?
   (b) Find a basis for the kernel of $f$. What is the nullity of $f$?

8. Define a homomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$h\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 + 2x_3 \\ -x_1 - 2x_1 + 2x_3 \\ 2x_1 + x_2 + x_3 \end{bmatrix}.$$

   (a) Is $f$ one-to-one? Explain.
   (b) Is $f$ onto? Explain.
   (c) If possible, find a vector $\vec{v}$ such that $f(\vec{v}) = \vec{b} = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$.

9. Let $\mathcal{B} = \langle p_1(x), p_2(x), p_3(x) \rangle$, where

$$p_1(x) = 1 + 2x, \quad p_2(x) = x - x^2, \quad p_3(x) = x + x^2,$$

and let $f : \mathcal{P}_2 \to \mathcal{P}_2$ be the homomorphism defined by $f(p(x)) = \frac{d}{dx}p(x) - p(x)$.
   (a) Show that $\mathcal{B}$ is a basis of $\mathcal{P}_2$.
   (b) If $q(x) = 1 + 3x + x^2$, compute $\mathrm{Rep}_{\mathcal{B}}(q(x))$, the representation of $q(x)$ with respect to the basis $\mathcal{B}$.
   (c) Determine, with justification, if the kernel of $f$ is $\{\vec{0}\}$.
   (d) Based on your answer to the previous part, is $f$ an isomorphism?

10. (a) Suppose that $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in a vector space $V$ and $f : V \to W$ is a homomorphism. If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are linearly dependent, it is true that $f(\vec{u})$, $f(\vec{v})$, and $f(\vec{w})$ are linearly dependent? Verify if true, provide a counterexample if false.

    (b) Suppose that $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in a vector space $V$ and $f : V \to W$ is a homomorphism. If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are linearly independent, it is true that $f(\vec{u})$, $f(\vec{v})$, and $f(\vec{w})$ are linearly independent? Verify if true, provide a counterexample if false.

    (c) Is the conclusion of either of the two previous parts different if the word *homomorphism* is replaced by the word *isomorphism*?

    (d) If $f : \mathbb{R}^6 \to \mathbb{R}^4$ is a homomorphism, is it possible that the null space of $f$ has dimension one?

11. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a homomorphism.

    (a) If $f$ is a one-to-one mapping, what is the dimension of the range of $f$? Explain why.

    (b) If $f$ is onto $\mathbb{R}^m$, what is the dimension of the kernel of $f$? Explain why.