Exam 1 will be on Monday, October 16, 2017. The syllabus for Exam 2 consists of Sections Two.III.1, Two.III.2, Two.III.3, Three.I, and Three.II. You should know the main definitions, results and computational techniques that we have covered in these sections. Of The following are problems similar to those you might expect on your exam. Problems can also be similar to assigned homework problems and suggested problems from the text, so you should certainly review those. You may, and are encouraged to, bring any questions that you have to be discussed during class on Friday, October 13.

1. Let $A = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 6 & 9 & 6 & 3 \\ 1 & 2 & 4 & 1 & 2 \\ 2 & 4 & 9 & 1 & 2 \end{bmatrix}$ and let $R = \begin{bmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. You may assume that R is the reduced row-echelon form of A, that is, $R = \text{rref}(A)$. Using this fact, answer the following questions.

   (a) Find a basis of the column space of A.

   (b) Find a basis of the row space of A.

   (c) Find a basis of the null space of A.

   (d) What is the rank of A?

▶ **Solution.** (a) The columns of R with leading ones are 1, 3, and 5. The corresponding columns of A for a basis for the column space of A. Thus, a basis is

$$\left\langle \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 4 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \\ 2 \end{bmatrix} \right\rangle.$$

(b) The nonzero rows of R form a basis of the row space. Thus, a basis is

$$\langle \begin{bmatrix} 1 & 2 & 0 & 5 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix} \rangle$$

(c) The null space of A is the same as the null space of R. R is the coefficient matrix of the homogeneous system

$$\begin{aligned} x_1 + 2x_2 \quad + 5x_4 \quad &= 0 \\ x_3 - \ x_4 \quad &= 0 \\ x_5 &= 0 \end{aligned}$$

The leading variables are x_1, x_3, and x_5, so the free variables are x_2 and x_4. Thus, the null space of A is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 5x_4 \\ x_2 \\ x_4 \\ x_4 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -5 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} x_4$$

and so a basis of the null space of A is

$$\left\langle \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\rangle.$$

(d) The rank of A is the dimension of the row space of A, which is also the dimension of the column space of A. Thus, the rank of A is 3.

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2. Which of the following sets of vectors are bases for $\mathbb{R}^2$?

(a) $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

(e) $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$

(f) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

▶ **Solution.** (a) and (d) are bases of $\mathbb{R}^2$ since each is clearly linearly independent (the second vector is not a scalar multiple of the first) and there are 2 vectors, which is the dimension of $\mathbb{R}^2$. (b) is not linearly independent, since it consists of 3 vectors in a vector space of dimension 2 (specifically, the third vector is the sum of the first 2), and therefore , it is not a basis. (c) and (e) are not linearly independent (the second vector is a scalar multiple of the first in each case) and hence neither is a basis. (f) is not a basis, since it consists of fewer vectors than the dimension of $\mathbb{R}^2$. ◀

3. A square matrix A is *symmetric* if for all indices i and j, the entry $a_{i,j}$ equals the entry $a_{j,i}$. The symmetric $n \times n$ matrices are denoted Sym_n. Sym_n is a subspace of $\mathcal{M}_{n\times n}$.

(a) Find a basis of Sym_2. What is the dimension?

(b) Find a basis of Sym_3. What is the dimension?

(c) Find a basis of Sym_n. What is the dimension?

▶ **Solution.** (a)

$$\mathrm{Sym}_2 = \left\{ \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \middle| \, a_{1,2} = a_{2,1} \right\} = \left\{ a_{1,1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_{1,2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + a_{2,2} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Therefore, a basis is the three matrices listed above, namely,

$$\left\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle$$

(b) A symmetric 3×3 matrix A has the following form:

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{1,2} & a_{2,2} & a_{2,3} \\ a_{1,3} & a_{2,3} & a_{3,3} \end{bmatrix}$$

This can be parametrized using the distinct variables in this expression, namely $a_{i,i}$ for $1 \le i \le 3$ and $a_{i,j}$ for $1 \le i < j \le 3$. This gives

$$\begin{aligned} A = a_{1,1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{1,2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{1,3} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\ + a_{2,2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{2,3} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + a_{3.3} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

The 6 listed matrices are linearly independent and hence form a basis of Sym_3. These can be defined in terms of the *unit matrices* $E_{i,j}$ which are defined as the $n \times n$ matrices which have all 0's except for a 1 in the i, j position.. In the expression of the symmetric matrix A above, the listed matrices are $E_{1,1}$, $E_{1,2} + E_{2,1}$, $E_{1,3} + E_{3,1}$, $E_{2,2}$, $E_{2,3} + E_{3,2}$, $E_{3,3}$. Thus, the dimension of Sym_3 is 6.

(c) Symmetric means that the entry $a_{i,j}$ is the same as the entry $a_{j,i}$. Thus symmetric matrices can be parametrized using only entries $a_{i,j}$ for $i \le j$:

$$\begin{array}{rrrrr} a_{1,1}E_{1,1} + a_{1,2}(E_{1,2} + E_{2,1}) & + a_{1,3}(E_{1,3} + E_{3,1}) & + \cdots & + a_{1,n}(E_{1,n} + E_{n,1}) \\ + \quad a_{2,2}E_{2,2} & + a_{2,3}(E_{2,3} + E_{3,2}) & + \cdots & + a_{2,n}(E_{2,n} + E_{n,2}) \\ & + \quad a_{3,3}E_{3,3} & + \cdots & + a_{3,n}(E_{3,n} + E_{n,3}) \\ & & \ddots & \\ & & + & a_{n,n}E_{n,n} \end{array}$$

We can count the number of parameters in each row and then add to get the dimension. The first row has n, the second row $n-1$, the third row $n-2$, down to the last row which has 1. Thus, the number of parameters is $1 + 2 + \cdots + n = n(n+1)/2$, and this is the dimension of Sym_n. The basis consists of the matrices $E_{i,i}$ for $1 \le i \le n$ and $E_{i,j} + E_{j,i}$ for $1 \le i < j \le n$.

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4. Let $V = \mathrm{Span}(\mathcal{B})$ where $\mathcal{B} = \langle \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \rangle$ with

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix}, \ \vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \ \vec{v}_4 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}.$$

Find a subset of $\mathcal{B}$ that is a basis of V. What is the dimension of V?

▶ **Solution.** Since we want a basis consisting of a subset of $\mathcal{B}$, write the vectors as the columns of a matrix A and then find a basis for the column space of A in the normal manner by finding the reduced row echelon form of A. The matrix A is

$$A = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 1 & 1 \end{bmatrix}$$

Apply Gauss-Jordan reduction to A to get the reduced echelon form of A is

$$R = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The rank of this matrix is 3, and the first 3 columns contain leading 1's. Thus, the first three columns of A form a basis for the column space of A, which is V. Thus, a basis of V is $\langle \vec{v}_1, \vec{v}_2, \vec{v}_3 \rangle$. ◀

5. Find a basis for the intersection of the two planes through the origin in $\mathbb{R}^3$:

$$x - y + 2z = 0, \quad x + 2y - z = 0.$$

▶ **Solution.** The intersection of the two planes is just the solution set of the homogeneous system consisting of the two equations. This is found by row reducing the coefficient matrix $A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$. Row reduce this matrix to get the reduced row echelon form of A: $R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$. From this we compute the solution set (equal to the null space of A) as:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} z \quad \text{where } z \text{ is arbitrary.}$$

Thus, the intersection of the two planes is the one-dimensional vector space (a line through the origin) with basis $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$. ◀

6. Suppose the null space of a 5×6 matrix A is 4-dimensional.

   (a) What is the dimension of the row space of A?

(b) For what value of k is the column space of A a subspace of $\mathbb{R}^k$?

(c) For what value of k is the null space of A a subspace of $\mathbb{R}^k$?

▶ **Solution.** (a) The dimension of the null space of A is 4, so the rank is $6 - 4 = 2$. The rank is the dimension of the row space, so that dimension is 2.

(b) $k = 5$

(c) $k = 6$.

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7. Consider the function $f : \mathbb{R}^3 \to \mathbb{R}^4$ given by

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 + x_3 \\ -3x_1 + 6x_2 + x_3 \\ -x_1 + 2x_2 \\ 2x_1 - 4x_2 + x_3 \end{bmatrix}.$$

(a) Find a basis for the range of f. What is the rank of f?

(b) Find a basis for the kernel of f. What is the nullity of f?

▶ **Solution.** (a) The range of f is the column space of the matrix of the coefficients of the variables x_1, x_2, x_3:

$$A = \begin{bmatrix} 1 & -2 & 1 \\ -3 & 6 & 1 \\ -1 & 2 & 0 \\ 2 & -4 & 1 \end{bmatrix}$$

The reduced row echelon form of A is the matrix

$$R = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & d \\ 0 & 0 & 0 \end{bmatrix}$$

From this, it follows that the first and third columns of A form a basis of the column space of A, which is also the range of the homomorphism f. Thus, a basis of the range of f is

$$\left\langle \begin{bmatrix} 1 \\ -3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\rangle$$

The rank of f is the dimension of the range, that is 2.

(b) The kernel of f is the null space of the matrix A. This can be read off from the matrix R as the span of the vector $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. Thus, $\vec{v}$ is a basis of the kernel of f. Hence the nullity of f is 1.

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8. Define a homomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$h\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 + 2x_3 \\ -x_1 - 2x_1 + 2x_3 \\ 2x_1 + x_2 + x_3 \end{bmatrix}.$$

(a) Is f one-to-one? Explain.

(b) Is f onto? Explain.

(c) If possible, find a vector $\vec{v}$ such that $f(\vec{v}) = \vec{b} = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$.

▶ **Solution.** (a) The kernel of f is the null space of matrix of the coefficients of the variables x_1, x_2, x_3:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & -2 & 2 \\ 2 & 1 & 1 \end{bmatrix}$$

The reduced row echelon form of A is the matrix

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The null space of R is $\{\vec{0}\}$ so the kernel of f is $\{\vec{0}\}$. Thus, f is one-to-one.

(b) By the rank-nullity theorem, the rank of f is $3 - \text{nullity}(f) = 3$. Hence, the dimension of the range of f is 3, so the range of f is all of $\mathbb{R}^3$ and f is onto.

(c) Solve the equation $A\vec{x} = \vec{b}$ to find $\vec{v}$. To do this, apply Gauss-Jordan reduction to the augmented matrix

$$\left[A \mid \vec{b}\right] = \left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ -1 & -2 & 2 & -2 \\ 2 & 1 & 1 & 3 \end{array}\right]$$

to get $\vec{v} = \begin{bmatrix} 8/3 \\ -4/3 \\ -1 \end{bmatrix}$.

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9. Let $\mathcal{B} = \langle p_1(x),\, p_2(x),\, p_3(x)\rangle$, where

$$p_1(x) = 1 + 2x, \quad p_2(x) = x - x^2, \quad p_3(x) = x + x^2,$$

and let $f : \mathcal{P}_2 \to \mathcal{P}_2$ be the homomorphism defined by $f(p(x)) = \frac{d}{dx}p(x) - p(x)$.

(a) Show that $\mathcal{B}$ is a basis of $\mathcal{P}_2$.

(b) If $q(x) = 1+3x+x^2$, compute $\mathrm{Rep}_{\mathcal{B}}(q(x))$, the representation of $q(x)$ with respect to the basis $\mathcal{B}$.

(c) Determine, with justification, if the kernel of f is $\{\vec{0}\}$.

(d) Based on your answer to the previous part, is f an isomorphism?

▶ **Solution.** (a) Since the dimension of $\mathcal{P}_2$ is 3, it is sufficient to show that $\mathcal{B}$ is linearly independent. To do this, suppose that there is a dependence relation $c_1p_1(x) + c_2p_2(x) + c_3p_3(x) = \vec{0}$. Expanding out the left hand side gives an equation

$$c_1 + (2c_1 + c_2 + c_3)x + (c_3 - c_2)x^2 = 0.$$

This gives the system of equations

$$\begin{aligned} c_1 \qquad\qquad &= 0 \\ 2c_1 + c_2 + c_3 &= 0 \\ - c_2 + c_3 &= 0 \end{aligned}$$

Solving this gives $c_1 = 0$, $c_2 = 0$, $c_3 = 0$. Thus $\mathcal{B}$ is linearly independent and hence a basis since it has 3 vectors in a three dimensional space.

(b) $q(x) = p_1(x) + p_3(x)$ (by inspection). Thus

$$\mathrm{Rep}_{\mathcal{B}}(q(x)) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

(c) Suppose $p(x) = a + bx + cx^2$ is in the kernel of f. This means $f(p(x)) = \frac{d}{dx}p(x) - p(x) = 0$. Thus,

$$(b + 2cx) - (a + bx + cx^2) = 0.$$

This means that $(b-a) + (2c-b)x - cx^2 = 0$ which means that $b - a = 0$, $2c - b = 0$, and $-c = 0$. Solving gives $a = 0$, $b = 0$, $c = 0$, which means that $p(x) = 0$, so that the kernel of f is $\{\vec{0}\}$.

(d) Part (c) shows that the nullity of f is 0. By the rank-nullity theorem, the rank of f is 3, so f is also onto. Hence f is an isomorphism.

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10. (a) Suppose that $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in a vector space V and $f : V \to W$ is a homomorphism. If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are linearly dependent, is it true that $f(\vec{u})$, $f(\vec{v})$, and $f(\vec{w})$ are linearly dependent? Verify if true, provide a counterexample if false.

(b) Suppose that $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in a vector space V and $f : V \to W$ is a homomorphism. If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are linearly independent, it is true that $f(\vec{u})$, $f(\vec{v})$, and $f(\vec{w})$ are linearly independent? Verify if true, provide a counterexample if false.

(c) Is the conclusion of either of the two previous parts different if the word *homomorphism* is replaced by the word *isomorphism*?

(d) If $f : \mathbb{R}^6 \to \mathbb{R}^4$ is a homomorphism, is it possible that the null space of f has dimension one?

▶ **Solution.** (a) If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are linearly dependent, then there is a nontrivial linear dependence relation $c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = \vec{0}$, where nontrivial means that at least one $c_i \neq 0$. Since f is a homomorphism, it follows that

$$c_1 f(\vec{u}) + c_2 f(\vec{v}) + c_3 f(\vec{w}) = f(c_1\vec{u} + c_2\vec{v} + c_3\vec{w}) = f(\vec{0}) = \vec{0}$$

so there is a nontrivial linear dependence relation among $f(\vec{u})$, $f(\vec{v})$, and $f(\vec{w})$, so they are also linearly dependent.

(b) This is false. For a concrete example, consider $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$. Then $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are the standard basis of $\mathbb{R}^3$ and hence linearly independent. However, $f(\vec{u}) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $f(\vec{v}) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $f(\vec{w}) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ are linearly dependent since the third vector is $\vec{0}$.

(c) Part (a) is true as before since any isomorphism is also a homomorphism. Part (b) is true for an isomorphism. To see this, suppose $\vec{u}$, $\vec{v}$, and $\vec{w}$ are linearly independent. If there is a linear dependence relation $c_1 f(\vec{u}) + c_2(\vec{v}) + c_3 f(\vec{w}) = \vec{0}$, then it follows that

$$\vec{0} = c_1 f(\vec{u}) + c_2 f(\vec{v}) + c_3 f(\vec{w}) = f(c_1\vec{u} + c_2\vec{v} + c_3\vec{w})$$

so $c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = \vec{0}$ since an isomorphism is one-to-one. Since $\vec{u}$, $\vec{v}$, and $\vec{w}$ are linearly independent, this means that $c_1 = 0$, $c_2 = 0$ and $c_3 = 0$, so $f(\vec{u})$, $f(\vec{v})$, and $f(\vec{w})$ are linearly independent.

(d) The rank of f is at most 4 since the rank is the dimension of the range space. By the rank-nullity theorem the rank plus the nullity must add up to the dimension of the domain, which is 6. So the nullity must be at least 2.

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11. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a homomorphism.

(a) If f is a one-to-one mapping, what is the dimension of the range of f? Explain why.

(b) If f is onto $\mathbb{R}^m$, what is the dimension of the kernel of f? Explain why.

▶ **Solution.** (a) If f is one-to-one then the kernel of f is $\{\vec{0}\}$ so the nullity of f is 0. By the rank-nullity theorem, the rank of f must be the dimension of the domain of f, that is n. So the dimension of the range of f is n.

(b) If f is onto then the range of f is $\mathbb{R}^m$ so the rank of f is the dimension of the range of f, which is m. By the rank-nullity theorem, $\text{rank}(f) + \dim(\ker(f)) = n$ so the dimension of the kernel of f is $n - m$.

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