Instructions. Answer each of the questions on your own paper, and be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper. There are 5 problems, with a total of 70 points possible. The points for each problem is listed in parentheses. There is an additional 10 point bonus problem.

(10) 1. Use these matrices for this exercise.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 2 \\ -1 & 0 \\ 1 & 5 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

Compute each of the following matrices, or explain why the combination is not valid.

(a) $AB + BA$ (b) $2AB + C^2$.

▶ **Solution.** (a) AB is a 2×2 matrix, while BA is 3×3. Since they are not the same size, $AB + BA$ is not defined.

(b)

$$\begin{aligned} 2AB + C^2 &= 2\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -1 & 0 \\ 1 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \\ &= 2\begin{bmatrix} -1 & -3 \\ 4 & 12 \end{bmatrix} + \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -5 \\ 9 & 26 \end{bmatrix} \end{aligned}$$

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(30) 2. Let $h : \mathcal{P}_2 \to \mathcal{M}_{2\times 2}$ be the homomorphism represented with respect to the basis $\mathcal{B} = \langle 1, x, x^2 \rangle$ of $\mathcal{P}_2$ and the basis

$$\mathcal{D} = \left(\vec{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

of $\mathcal{M}_{2\times 2}$ by the matrix

$$H = \mathrm{Rep}_{\mathcal{B},\mathcal{D}}(h) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & -2 \\ 2 & 0 & 4 \\ 0 & 1 & -1 \end{bmatrix}.$$

(a) Find $h(\vec{v})$ for $\vec{v} = 2 + x - x^2$.

▶ **Solution.** $\mathrm{Rep}_{\mathcal{B}}(\vec{v}) = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ so that

$$\mathrm{Rep}_{\mathcal{D}}(h(\vec{v})) = \mathrm{Rep}_{\mathcal{B},\mathcal{D}}(h)\mathrm{Rep}_{\mathcal{B}}(\vec{v}) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & -2 \\ 2 & 0 & 4 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 0 \\ 2 \end{bmatrix}.$$

Thus, $h(\vec{v}) = \begin{bmatrix} 0 & 4 \\ 0 & 2 \end{bmatrix}$. ◀

(b) Find $h(\vec{w})$ for the general vector $\vec{w} = a + bx + cx^2$ in $\mathcal{P}_2$.

▶ **Solution.** $\mathrm{Rep}_{\mathcal{B}}(\vec{w}) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ so that

$$\mathrm{Rep}_{\mathcal{D}}(h(\vec{w})) = \mathrm{Rep}_{\mathcal{B},\mathcal{D}}(h)\mathrm{Rep}_{\mathcal{B}}(\vec{w}) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & -2 \\ 2 & 0 & 4 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+2c \\ 2b-2c \\ 2a+4c \\ b-c \end{bmatrix}.$$

Thus, $h(\vec{w}) = \begin{bmatrix} a+2c & 2b-2c \\ 2a+4c & b-c \end{bmatrix}$. ◀

(c) Find a basis for the range of h. What is the rank of h?

▶ **Solution.** From the previous part, for $\vec{w} = a + bx + cx^2 \in \mathcal{P}_2$,

$$h(\vec{w}) = \begin{bmatrix} a+2c & 2b-2c \\ 2a+4c & b-c \end{bmatrix} = (a+2c)\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} + (b-c)\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}.$$

Thus, the range of h is $\mathrm{Span}\langle \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \rangle$. Since $\langle \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \rangle$ is linearly independent, it is a basis of the range of h.
The rank of h is the dimension of the range of h, which is 2. ◀

(d) Is $A = \begin{bmatrix} 3 & -4 \\ 6 & 2 \end{bmatrix}$ in the range of h? What about $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$?

▶ **Solution.** For A to be in the range of h, it is necessary to be able to write A as a linear combination of the basis vectors for range h:

$$A = c_1 \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} c_1 & 2c_2 \\ 2c_1 & c_2 \end{bmatrix}.$$

This gives $c_1 = 3$, $2c_2 = -4$, $2c_1 = 6$, $c_2 = 2$. This gives a contradiction $c_2 = -2$ and $c_2 = 2$, so A is not in the range of h.
For B the similar equations would be $c_1 = 2$, $c_2 = 1$, $2c_1 = 1$, $c_2 = 2$. These equations have no solutions, so B is not in the range of h. ◀

(e) Find a basis for the null space of h. What is the nullity of h?

▶ **Solution.** The null space of h corresponds under the basis representation map to the null space of the matrix H. Row reduce H to get the row reduced echelon form of H:

$$\operatorname{rref}(H) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The null space of H is thus the solution set to the system

$$\begin{aligned} c_1 + \phantom{c_2 -{}} 2c_3 &= 0 \\ c_2 - c_3 &= 0 \end{aligned}$$

There is one free variable c_3 for this system and the general solution is

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -2c_3 \\ c_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} c_3 \quad c_3 \text{ arbitrary}.$$

Thus, $\left\langle \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\rangle$ is a basis for the null space of H. This corresponds to $\langle -2+x+x^2 \rangle$ as a basis for the null space of h. The nullity of h is 1 since the null space has dimension 1. ◀

(f) Is $p_1(x) = 1 + x - 2x^2$ in the null space of h? What about $p_2(x) = 2 - x - x^2$?

▶ **Solution.** $p_2(x) = (-1)(-2+x+x^2)$ so $p_2(x)$ is in the null space of h. However, p_2 is not a scalar multiple of $-2 + x + x^2$, so it is not in the null space of h. ◀

(10) 3. Let $A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{bmatrix}$. Compute the determinant of A. Be sure to show all of your work.

▶ **Solution.** There are several methods available. One is to use elementary row operations to reduce the matrix to an upper triangular matrix. Alternatively, use Laplace expansion. In this case the third column has only one nonzero entry, so expand along that column, and then expand the resulting 3×3 matrix along the second column:

$$|A| = (-1)(4) \begin{vmatrix} 1 & 0 & 2 \\ 5 & 4 & 3 \\ 2 & 0 & 1 \end{vmatrix} = (-4)(4) \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = (-4)(4)(-3) = 48.$$

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(10) 4. Find the inverse, if possible, of the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & -2 \\ 0 & 1 & 1 \end{bmatrix}$.

▶ **Solution.** Row reduce the augmented matrix $\left[A \mid I_3\right]$ to produce $\left[I_3 \mid A^{-1}\right]$, if possible.

$$\left[A \mid I_3\right] \xrightarrow[\rho_2+\rho_3]{} \xrightarrow[(-1)\rho_3]{(-1)\rho_2} \xrightarrow[-2\rho_3+\rho_2]{-\rho_3+\rho_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{array}\right].$$

Thus,

$$A^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & -1 & -1 \end{bmatrix}.$$

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(10) 5. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. If $\det A = 5$, compute the determinant of each of the following matrices.

(a) $B = -2A^2$ (b) $C = \begin{bmatrix} a & b & c \\ 2d-3g & 2e-3h & 2f-3i \\ g & h & i \end{bmatrix}$.

▶ **Solution.** (a) $|B| = (-2)^3|A|^2 = (-8)(25) = -200$.

(b) Adding 3 times row 3 to row 2 does not change the determinant, so

$$|C| = \begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{vmatrix} = 2|A| = 2 \cdot 5 = 10.$$

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Bonus. Let A and B be 2×2 matrices with $AB = 0$. Determine if each of the following statements is **True** or **False**. If True, explain why. If False, provide a counterexample.

(a) $A = 0$ or $B = 0$.

(b) If A is invertible, then $B = 0$.

(c) $BA = 0$.

▶ **Solution.** (a) **False**. For a counterexample, let $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and let $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $AB = 0$, but $A \neq 0$ and $B \neq 0$.

(b) **True** If $AB = 0$ and A is invertible, then $B = I_2B = (A^{-1}A)B = A^{-1}(AB) = A^{-1}0 = 0$.

(c) **False** The counterexample from part (a) also works here: $AB = 0$ but $BA = B \neq 0$.

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