Exam 3 will be on Wednesday November 15, 2017. The syllabus for Exam 3 consists of Sections Three.III, Three.IV, Three.V.1, Four.I and Four.iii. You should know the main definitions, results and computational techniques that we have covered in these sections. Of The following are problems similar to those you might expect on your exam. Problems can also be similar to assigned homework problems and suggested problems from the text, so you should certainly review those. You may, and are encouraged to, bring any questions that you have to be discussed during class on Monday, November 13.

1. Let $T$ represent a map $t\colon \mathbb{R}^2 \to \mathbb{R}^2$ with respect to the standard bases.

$$T = \begin{bmatrix} 2 & -1 \\ 6 & -3 \end{bmatrix}$$

Find the range space and null space of $t$.

▶ **Solution.** The homomorphism $t$ acts on $\mathbb{R}^2$ as multiplication by the matrix $T$. Thus, the range of $t$ is the column space of $T$, which is $\operatorname{span}(\begin{bmatrix} 1 \\ 3 \end{bmatrix})$. The null space is $\operatorname{span}(\begin{bmatrix} 1 \\ 2 \end{bmatrix})$. ◀

2. Find the determinant.

$$\begin{vmatrix} 1 & 0 & 2 & 0 \\ 2 & 1 & 1 & 1 \\ 0 & -1 & 1 & -1 \\ 2 & 2 & 1 & 0 \end{vmatrix}$$

▶ **Solution.** Call the matrix $A$. Then $A$ is reduced to an upper triangular matrix by the following sequence of elementary row operations:

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 2 & 1 & 1 & 1 \\ 0 & -1 & 1 & -1 \\ 2 & 2 & 1 & 0 \end{bmatrix} \xrightarrow[-2\rho_1+\rho_4]{-2\rho_1+\rho_2} \xrightarrow[\rho_2+\rho_3]{-2\rho_2+\rho_4} \xrightarrow[\frac{3}{2}\rho_3+\rho_4]{} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} = B$$

Since each of these elementary row operations does not change the determinant, it follows that the determinant of the original matrix $A$ is the determinant of the upper triangular matrix $B$. The latter determinant is the product of the diagonal entries. Thus

$$\det A = \det B = 4.$$

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3. Represent the linear map $d/dx\colon \mathcal{P}_4 \to \mathcal{P}_3$ with respect to $\mathcal{B} = \langle 1, x, x^2, x^3, x^4\rangle$ and $\mathcal{D} = \langle 1, 1+x, 1+x+x^2, 1+x+x^2+x^3\rangle$.

▶ **Solution.** Done in class. Answer:

$$\mathrm{Rep}_{\mathcal{B},\mathcal{D}}\left(\frac{d}{dx}\right) = \begin{bmatrix} 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 2 & -3 & 0 \\ 0 & 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

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4. Let $T : \mathcal{M}_{2\times 2} \to \mathcal{M}_{2\times 2}$ be defined by $T(A) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} A$. Using the basis

$$\mathcal{B} = \left(\vec{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 & 0) \\ 1 & 0 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

for $\mathcal{M}_{2\times 2}$, find the following.

(a) The matrix $\mathrm{Rep}_{\mathcal{B},\mathcal{B}}(T)$ representing $T$ with respect to the bases $\mathcal{B}$ on the domain and $\mathcal{B}$ on the codomain.

(b) A basis for the null space of $T$.

(c) A basis for the range of $T$.

▶ **Solution.** (a)

$$\begin{aligned}
\mathrm{Rep}_{\mathcal{B},\mathcal{B}}(T) &= \begin{bmatrix} \mathrm{Rep}_{\mathcal{B}}(T(\vec{v}_1) & \mathrm{Rep}_{\mathcal{B}}(T(\vec{v}_2) & \mathrm{Rep}_{\mathcal{B}}(T(\vec{v}_3) & \mathrm{Rep}_{\mathcal{B}}(T(\vec{v}_4) \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{Rep}_{\mathcal{B}}(\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}) & \mathrm{Rep}_{\mathcal{B}}(\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}) & \mathrm{Rep}_{\mathcal{B}}(\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}) & \mathrm{Rep}_{\mathcal{B}}(\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}) \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}
\end{aligned}$$

(b) First find a basis for the null space of the matrix representation $\mathrm{Rep}_{\mathcal{B},\mathcal{B}}(T)$. Row reduce this matrix to get

$$R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis for the null space of the matrix $R$ is $< \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} >$, and this corresponds to the following basis of the null space of $T$: $< \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} >$

(c) A basis for the column space of $\mathrm{Rep}_{\mathcal{B},\mathcal{B}}(T)$ is $< \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} >$, and this corresponds to the following basis of the range space of $T$: $< \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} >$ ◀

5. Let $L : \mathcal{M}_{2\times 2} \to \mathcal{M}_{2\times 2}$ be defined by $L(A) = A^T$, the *transpose* of the matrix $A$. Using the basis

$$\mathcal{B} = \left( \vec{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 & 0) \\ 1 & 0 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

for $\mathcal{M}_{2\times 2}$, find the matrix representation of $L$ using the basis $\mathcal{B}$ on both the domain and codomain.

▶ **Solution.** Done in class. ◀

6. Let $T : P_2 \to P_2$ be defined by $T(f(x)) = 2f''(x) - f(x)$. Use $\mathcal{B} = \langle 1, x, x^2 \rangle$ as a basis for $P_2$.

   (a) Find the matrix $\mathrm{Rep}_{\mathcal{B},\mathcal{B}}(T)$ of $T$ with respect to the basis $\mathcal{B}$ on both the domain and codomain.

   (b) Find $\det(\mathrm{Rep}_{\mathcal{B},\mathcal{B}}(T))$.

   (c) Is $T$ an isomorphism? Why or why not?

▶ **Solution.** (a) $T(1) = -1$, $T(x) = -x$, $T(x^2) = 4 - x^2$. Thus,

$$\mathrm{Rep}_{\mathcal{B},\mathcal{B}}(T) = \begin{bmatrix} -1 & 0 & 4 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

(b) $\det(\mathrm{Rep}_{\mathcal{B},\mathcal{B}}(T)) = -1$ (product of the diagonal entries since the matrix is upper triangular).

(c) $T$ is an isomorphism since $\det(\mathrm{Rep}_{\mathcal{B},\mathcal{B}}(T)) \neq 0$.

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7. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{bmatrix}$. Evaluate $\det A$ using row-reduction, showing all of your work.

▶ **Solution.** Call the matrix $A$. Then $A$ is reduced to an upper triangular matrix by the following sequence of elementary row operations:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{bmatrix} \xrightarrow[\substack{-3\rho_1+\rho_3 \\ -2\rho_1+\rho_4}]{-4\rho_1+\rho_2} \xrightarrow{\rho_2\leftrightarrow\rho_4} \xrightarrow[-2\rho_2+\rho_3]{-7\rho_2+\rho_4} \xrightarrow[\rho_3+\rho_4]{} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -7 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & 0 & 40 \end{bmatrix} = B$$

Since each of these elementary row operations does not change the determinant, except for the single row swap, which multiplies the determinant by $-1$, it follows that the determinant of the original matrix $A$ is $-1\times$ the determinant of the upper triangular matrix $B$. The latter determinant is the product of the diagonal entries. Thus

$$\det A = -\det B = -160.$$

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8. Let $A$ be the $3\times 3$ matrix expressed in rows as $A = \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{bmatrix}$, and let $B$ be described in terms of the rows of $A$ as $B = \begin{bmatrix} \mathbf{R}_1+\mathbf{R}_2+\mathbf{R}_3 \\ 2\mathbf{R}_3 \\ 3\mathbf{R}_2+\mathbf{R}_3 \end{bmatrix}$. If $\det A = 5$, find $\det B$.

▶ **Solution.** Use elementary row operations to transform $B$ into $A$, keeping track of the effect of each on the determinant.

$$|B| = \begin{vmatrix} \mathbf{R}_1+\mathbf{R}_2+\mathbf{R}_3 \\ 2\mathbf{R}_3 \\ 3\mathbf{R}_2+\mathbf{R}_3 \end{vmatrix} = 2\begin{vmatrix} \mathbf{R}_1+\mathbf{R}_2+\mathbf{R}_3 \\ \mathbf{R}_3 \\ 3\mathbf{R}_2+\mathbf{R}_3 \end{vmatrix} = 2\begin{vmatrix} \mathbf{R}_1+\mathbf{R}_2 \\ \mathbf{R}_3 \\ 3\mathbf{R}_2 \end{vmatrix} = 2\cdot 3\begin{vmatrix} \mathbf{R}_1+\mathbf{R}_2 \\ \mathbf{R}_3 \\ \mathbf{R}_2 \end{vmatrix}$$

$$= 2\cdot 3\begin{vmatrix} \mathbf{R}_1 \\ \mathbf{R}_3 \\ \mathbf{R}_2 \end{vmatrix} = 2\cdot 3\cdot(-1)\begin{vmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{vmatrix} = 2\cdot 3\cdot(-1)|A| = 2\cdot 3\cdot(-1)\cdot 5 = -30.$$

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9. Let $\mathcal{B} = \left(\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)$ be a basis of $R^2$. Let $A = \begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix}$ and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by multiplication by $A$: $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^2$. Compute the matrix $\operatorname{Rep}_{\mathcal{B},\mathcal{B}}(T)$ representing $T$ with respect to the bases $\mathcal{B}$ on the domain and $\mathcal{B}$ on the codomain.

▶ **Solution.** $T(\vec{v}_1) = A\vec{v}_1 = \begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} = -12\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 9\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $T(\vec{v}_2) = A\vec{v}_2 = \begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \end{bmatrix} = -18\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 13\begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Thus,

$$\operatorname{Rep}_{\mathcal{B},\mathcal{B}}(T) = \begin{bmatrix} -12 & -18 \\ 9 & 13 \end{bmatrix}.$$

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10. Determine if each of the following statements is True or False. Explain why.

    (a) Let $V$ be the space of all functions from $\mathbb{R}$ to $\mathbb{R}$ that have infinitely many derivatives. The function $T : V \to V$ given by $T(v) = 3f' - 2f''$ is a linear transformation. **True**

    (b) If the determinant of a $4 \times 4$ matrix is 4, then the rank of the matrix must be 4. **True**

    (c) $\det(A + B) = \det A + \det B$. **False** Counterexample: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

    (d) $\det(kA) = k \det A$. **False** The correct formula is $\det(kA) = k^n \det A$ if $A$ is $n \times n$.

    (e) $\det AB = \det A \det B$. **True**

    (f) If all entries of the diagonal of a matrix $A$ are 0, then $\det A = 0$. **False** Counterexample: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

    (g) $(AB)^{-1} = A^{-1}B^{-1}$ **False** The order is wrong. It should be $(AB)^{-1} = B^{-1}A^{-1}$

11. If $A = \begin{bmatrix} 1 & 2 & -4 \\ -1 & -1 & 5 \\ 2 & 7 & -3 \end{bmatrix}$, find $A^{-1}$.

▶ **Solution.** Use row operations on the augmented matrix $[A \mid I_3]$.

$$[A \mid I_3] \xrightarrow[-2\rho_1+\rho_3]{\rho_1+\rho_2} \xrightarrow[-3\rho_2+\rho_3]{-2\rho_2+\rho_3} \xrightarrow{\frac{1}{2}} \xrightarrow[-\rho_3+\rho_2]{-6\rho_3+\rho_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -16 & -11 & 3 \\ 0 & 1 & 0 & 7/2 & 5/2 & -1/2 \\ 0 & 0 & 1 & -5/2 & -3/2 & 1/2 \end{array}\right].$$

Thus,

$$A^{-1} = \begin{bmatrix} -16 & -11 & 3 \\ 7/2 & 5/2 & -1/2 \\ -5/2 & -3/2 & 1/2 \end{bmatrix}.$$

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12. Let $A = \begin{bmatrix} 2 & 3 & 0 & 2 \\ 4 & 3 & 2 & 1 \\ 8 & 3 & 0 & 5 \\ 7 & 0 & 0 & 4 \end{bmatrix}$. Use Laplace's cofactor expansion to compute $\det A$.

▶ **Solution.** Use cofactor expansion along column 3:

$$|A| = -2 \begin{vmatrix} 2 & 3 & 2 \\ 8 & 3 & 5 \\ 7 & 0 & 4 \end{vmatrix} = -2 \left( 7 \begin{vmatrix} 3 & 2 \\ 3 & 5 \end{vmatrix} + 4 \begin{vmatrix} 2 & 3 \\ 8 & 3 \end{vmatrix} \right)$$
$$= -2(7(15 - 6) + 4(6 - 24)) = 18.$$

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13. Find all values of $\lambda$ for which the matrix $\begin{bmatrix} \lambda & 1 & -1 \\ 1 & 2 & -2 \\ -1 & 1 & 0 \end{bmatrix}$ is not invertible.

▶ **Solution.** The matrix is not invertible precisely when $|A| = 0$. Compute the determinant (using Laplace expansion along row 3 for example) to get $|A| = -(2\lambda + 1)$. So $A$ is not invertible only if $\lambda = 1/2$. ◀