In all homework problems, it is not sufficient to show only the answers. *You must show your work.* These exercises are based on Chapter Five.II from the text.

For each of the matrices below

(a) calculate the characteristic polynomial of A,

(b) find the eigenvalues of A,

(c) find a basis for each eigenspace of A,

(d) determine whether or not A is diagonalizable. If A is diagonalizable, then find an invertible matrix P and a diagonal matrix D such that $P^1AP = D$.

1. $A = \begin{bmatrix} -1 & -2 \\ 6 & 6 \end{bmatrix}$

▶ **Solution.** (a) The characteristic polynomial of A is

$$\begin{aligned}\det(A-\lambda I) &= \begin{vmatrix} -1-\lambda & -2 \\ 6 & 6-\lambda \end{vmatrix} \\ &= (-1-\lambda)(6-\lambda) - (-2)6 \\ &= \lambda^2 - 5\lambda + 6 \\ &= (\lambda-2)(\lambda-3).\end{aligned}$$

(b) A has eigenvalues 2 and 3, each with algebraic multiplicity 1.

(c) The eigenspace of A associated to the eigenvalue 2 is the null space of the matrix $A - 2I$. To find a basis for the eigenspace, row reduce this matrix.

$$A - 2I = \begin{bmatrix} -3 & -2 \\ 6 & 4 \end{bmatrix} \xrightarrow[-\frac{1}{3}\rho_1]{2\rho_1+\rho_2} \begin{bmatrix} 1 & \frac{2}{3} \\ 0 & 0 \end{bmatrix}$$

Thus, the general solution to the equation $(A-2I)\vec{x} = \vec{0}$ is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix} x_2$ where x_2 is arbitrary. Letting $x_2 = 3$ gives $\mathcal{B}_2 = \langle \begin{bmatrix} -2 \\ 3 \end{bmatrix} \rangle$ as a basis of the eigenspace associated to the eigenvalue 2. (Any nonzero x_2 can be used. 3 is chosen to not have a fraction.)

The eigenspace of A associated to the eigenvalue 3 is the null space of the matrix $A - 3I$. To find a basis for the eigenspace, row reduce this matrix.

$$A - 3I = \begin{bmatrix} -4 & -2 \\ 6 & 3 \end{bmatrix} \xrightarrow[-6\rho_1+\rho_2]{-\frac{1}{4}\rho_1} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

Thus, the general solution to the equation $(A-3I)\vec{x}=\vec{0}$ is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} x_2$ where x_2 is arbitrary. Letting $x_2 = 2$ gives $\mathcal{B}_3 = \langle \begin{bmatrix} -1 \\ 2 \end{bmatrix} \rangle$ as a basis of the eigenspace associated to the eigenvalue 3.

(d) A is diagonalizable since there is a basis of $\mathbb{R}^2$ consisting of eigenvectors of A. Specifically, concatenate $\mathcal{B}_2$ and $\mathcal{B}_3$ to get such a basis $\mathcal{B} = \langle \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix} \rangle$. If we set $P = \begin{bmatrix} -2 & -1 \\ 3 & 2 \end{bmatrix}$, then P is invertible and

$$P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

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2. $A = \begin{bmatrix} 11 & 25 \\ -4 & -9 \end{bmatrix}$

▶ **Solution.** (a) The characteristic polynomial of A is

$$\begin{aligned} \det(A-\lambda I) &= \begin{vmatrix} 11-\lambda & 25 \\ -4 & -9-\lambda \end{vmatrix} \\ &= (11-\lambda)(-9-\lambda)-(25)(-4) \\ &= \lambda^2 - 2\lambda + 1 \\ &= (\lambda-1)^2 \end{aligned}$$

(b) A has eigenvalue 1 with algebraic multiplicity 2.

(c) The eigenspace of A associated to the eigenvalue 1 is the null space of the matrix $A-I$. To find a basis for the eigenspace, row reduce this matrix.

$$A - I = \begin{bmatrix} 10 & 25 \\ -4 & -10 \end{bmatrix} \xrightarrow[4\rho_1+\rho_2]{\frac{1}{10}\rho_1} \begin{bmatrix} 1 & \frac{5}{2} \\ 0 & 0 \end{bmatrix}$$

Thus, the general solution to the equation $(A-I)\vec{x}=\vec{0}$ is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} \\ 1 \end{bmatrix} x_2$ where x_2 is arbitrary. Letting $x_2 = 2$ gives $\mathcal{B}_1 = \langle \begin{bmatrix} -5 \\ 2 \end{bmatrix} \rangle$ as a basis of the eigenspace associated to the eigenvalue 1.

(d) A is not diagonalizable since there is only a one dimensional eigenspace associated to the only eigenvalue.

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3. $A = \begin{bmatrix} -1 & -3 & -3 \\ 3 & 5 & 3 \\ -1 & -1 & 1 \end{bmatrix}$

▶ **Solution.** (a) The characteristic polynomial of A is

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} -1-\lambda & -3 & -3 \\ 3 & 5-\lambda & 3 \\ -1 & -1 & 1-\lambda \end{vmatrix} = (-1)\begin{vmatrix} 1+\lambda & 3 & 3 \\ 3 & 5-\lambda & 3 \\ -1 & -1 & 1-\lambda \end{vmatrix} \\
&= (-1)\begin{vmatrix} 0 & 2-\lambda & 4-\lambda^2 \\ 0 & 2-\lambda & 6-3\lambda \\ -1 & -1 & 1-\lambda \end{vmatrix} \\
&= (-1)(-1)\begin{vmatrix} 2-\lambda & 4-\lambda^2 \\ 2-\lambda & 6-3\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 1 & 4-\lambda^2 \\ 1 & 6-3\lambda \end{vmatrix} \\
&= (2-\lambda)(6-3\lambda-(4-\lambda^2)) = (2-\lambda)(\lambda^2-3\lambda+2) \\
&= -(\lambda-1)(\lambda-2)^2.
\end{aligned}$$

(b) A has eigenvalues 1 and 2, with algebraic multiplicities 1 and 2 respectively.

(c) The eigenspace of A associated to the eigenvalue 1 is the null space of the matrix $A - I$. To find a basis for the eigenspace, row reduce this matrix.

$$A - I = \begin{bmatrix} -2 & -3 & -3 \\ 3 & 4 & 3 \\ -1 & -1 & 0 \end{bmatrix} \longrightarrow \cdots \longrightarrow \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the general solution to the equation $(A-I)\vec{x} = \vec{0}$ is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} x_3$ where x_3 is arbitrary. Letting $x_3 = 1$ gives $\mathcal{B}_1 = \langle \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} \rangle$ as a basis of the eigenspace associated to the eigenvalue 1.

The eigenspace of A associated to the eigenvalue 2 is the null space of the matrix $A - 2I$. To find a basis for the eigenspace, row reduce this matrix.

$$A - 2I = \begin{bmatrix} -3 & -3 & -3 \\ 3 & 3 & 3 \\ -1 & -1 & -1 \end{bmatrix} \longrightarrow \cdots \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the general solution to the equation $(A-2I)\vec{x} = \vec{0}$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} x_3$$

where x_2 and x_3 are arbitrary. Thus $\mathcal{B}_2 = \langle \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \rangle$ as a basis of the eigenspace associated to the eigenvalue 2.

(d) A is diagonalizable since there is a basis of $\mathbb{R}^3$ consisting of eigenvectors of A. Specifically, concatenate $\mathcal{B}_1$ and $\mathcal{B}_2$ to get such a basis

$$\mathcal{B} = \langle \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \rangle.$$

If we set

$$P = \begin{bmatrix} 3 & -1 & -1 \\ -3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

, then P is invertible and

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

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4. $A = \begin{bmatrix} 2 & 5 & 10 \\ 1 & 2 & 4 \\ -1 & -1 & -4 \end{bmatrix}$

▶ **Solution.** (a) The characteristic polynomial of A is

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} 2-\lambda & 5 & 10 \\ 1 & 2-\lambda & 4 \\ -1 & -1 & -4-\lambda \end{vmatrix} \\
&= \begin{vmatrix} 0 & 3+\lambda & \lambda^2+2\lambda+2 \\ 0 & 1-\lambda & -\lambda \\ -1 & -1 & -4-\lambda \end{vmatrix} \\
&= (-1)\begin{vmatrix} 3+\lambda & \lambda^2+2\lambda+2 \\ 1-\lambda & -\lambda \end{vmatrix} \\
&= (-1)(-\lambda^2 - 3\lambda + (\lambda-1)(\lambda^2+2\lambda+2)) \\
&= (-1)(\lambda^3 - 3\lambda - 2) \\
&= -(\lambda+1)^2(\lambda-2).
\end{aligned}$$

(b) A has eigenvalues -1 and 2, with algebraic multiplicities 2 and 1 respectively.

(c) The eigenspace of A associated to the eigenvalue -1 is the null space of the matrix $A-(-1)I = A+I$. To find a basis for the eigenspace, row reduce this matrix.

$$A+I = \begin{bmatrix} 3 & 5 & 10 \\ 1 & 3 & 4 \\ -1 & -1 & -3 \end{bmatrix} \longrightarrow \cdots \longrightarrow \begin{bmatrix} 1 & 0 & \frac{5}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the general solution to the equation $(A+I)\vec{x} = \vec{0}$ is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} x_3$ where x_3 is arbitrary. Letting $x_3 = 2$ gives $\mathcal{B}_{-1} = \langle \begin{bmatrix} -5 \\ -1 \\ 2 \end{bmatrix} \rangle$ as a basis of the eigenspace associated to the eigenvalue -1.

The eigenspace of A associated to the eigenvalue 2 is the null space of the matrix $A-2I$. To find a basis for the eigenspace, row reduce this matrix.

$$A-2I = \begin{bmatrix} 0 & 5 & 10 \\ 1 & 0 & 4 \\ -1 & -1 & -6 \end{bmatrix} \longrightarrow \cdots \longrightarrow \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the general solution to the equation $(A-2I)\vec{x} = \vec{0}$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 1 \end{bmatrix} x_3$$

where x_3 is arbitrary. Thus $\mathcal{B}_2 = \langle \begin{bmatrix} -4 \\ -2 \\ 1 \end{bmatrix} \rangle$ as a basis of the eigenspace associated to the eigenvalue 2.

(d) A is not diagonalizable since the sum of the dimensions of the eigenspaces is $1+1 < 3 = \dim \mathbb{R}^3$.

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5. $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 5 & 5 \\ 0 & 0 & -1 \end{bmatrix}$

▶ **Solution.** (a) The characteristic polynomial of A is

$$\begin{aligned} \det(A-\lambda I) &= \begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 5-\lambda & 5 \\ 0 & 0 & -1-\lambda \end{vmatrix} \\ &= (-1-\lambda)\begin{vmatrix} 1-\lambda & 0 \\ 1 & 5-\lambda \end{vmatrix} \\ &= -(\lambda+1)(1-\lambda)(5-\lambda). \end{aligned}$$

(b) A has eigenvalues -1, 1 and 5, each with algebraic multiplicity 1.

(c) The eigenspace of A associated to the eigenvalue -1 is the null space of the matrix $A-(-1)I = A+I$. To find a basis for the eigenspace, row reduce this matrix.

$$A+I = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 6 & 5 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \cdots \longrightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{11}{12} \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the general solution to the equation $(A+I)\vec{x} = \vec{0}$ is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{11}{12} \\ 1 \end{bmatrix} x_3$ where x_3 is arbitrary. Letting $x_3 = 12$ gives $\mathcal{B}_{-1} = \langle \begin{bmatrix} 6 \\ -11 \\ 12 \end{bmatrix} \rangle$ as a basis of the eigenspace associated to the eigenvalue -1.

The eigenspace of A associated to the eigenvalue 1 is the null space of the matrix $A-I$. To find a basis for the eigenspace, row reduce this matrix.

$$A-I = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 4 & 5 \\ 0 & 0 & -2 \end{bmatrix} \longrightarrow \cdots \longrightarrow \begin{bmatrix} 1 & 4 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the general solution to the equation $(A-I)\vec{x} = \vec{0}$ is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} x_2$ where x_2 is arbitrary. Letting $x_2 = 1$ gives $\mathcal{B}_1 = \langle \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} \rangle$ as a basis of the eigenspace associated to the eigenvalue 1.

The eigenspace of A associated to the eigenvalue 5 is the null space of the matrix $A-5I$. To find a basis for the eigenspace, row reduce this matrix.

$$A-5I = \begin{bmatrix} -4 & 0 & -1 \\ 1 & 0 & 5 \\ 0 & 0 & -6 \end{bmatrix} \longrightarrow \cdots \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the general solution to the equation $(A-5I)\vec{x} = \vec{0}$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} x_2$$

where x_2 is arbitrary. Thus $\mathcal{B}_5 = \langle \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rangle$ is a basis of the eigenspace associated to the eigenvalue 5.

(d) A is diagonalizable since there is a basis of $\mathbb{R}^3$ consisting of eigenvectors of A. Specifically, concatenate $\mathcal{B}_{-1}$, $\mathcal{B}_1$ and $\mathcal{B}_2$ to get such a basis

$$\mathcal{B} = \langle \begin{bmatrix} 6 \\ -11 \\ 12 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rangle.$$

If we set

$$P = \begin{bmatrix} 6 & -4 & 0 \\ -11 & 1 & 1 \\ 12 & 0 & 0 \end{bmatrix}$$

, then P is invertible and

$$P^{-1}AP = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

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