In all homework problems, it is not sufficient to show only the answers. *You must show your work.*

1. Find the solution set of each system.

(a)
$$\begin{aligned} x - 2y + z &= 1 \\ 2x + y - z &= 2 \\ 5x - 3y + \quad &= 0 \end{aligned}$$

▶ **Solution.** Apply Gauss's method to reduce the system to echelon form:

$$\xrightarrow[-5\rho_1+\rho_3]{-2\rho_1+\rho_2} \begin{aligned} x - 2y + z &= 1 \\ 5y - 3z &= 0 \\ 7y - 5z &= -5 \end{aligned} \quad \xrightarrow{-(7/5)\rho_2+\rho_3} \begin{aligned} x - 2y + z &= 1 \\ 5y - 3z &= 0 \\ -\tfrac{4}{5}z &= -5 \end{aligned}$$

Now use back substitution to get: $z = \frac{25}{4}$; $y = \frac{3}{5}z = \frac{15}{4}$; $x = 2y - z + 1 = 2(\frac{15}{4}) - \frac{25}{4} + 1 = \frac{9}{4}$. Thus, the unique solution is

$$(x, y, z) = (9/4, 15/4, 25/4).$$

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(b)
$$\begin{aligned} x + y + 2z &= 0 \\ x - 3y &= 7 \\ 3x + y + 7z &= 0 \\ 2y + z &= -3 \end{aligned}$$

▶ **Solution.** Apply Gauss's method to reduce the system to echelon form:

$$\xrightarrow[-3\rho_1+\rho_3]{-\rho_1+\rho_2} \begin{aligned} x + y + 2z &= 0 \\ -4y - 2z &= 7 \\ -2y + z &= 0 \\ 2y + z &= -3 \end{aligned} \quad \xrightarrow[(1/2)\rho_2+\rho_4]{-(1/2)\rho_2+\rho_3} \begin{aligned} x + y + 2z &= 0 \\ -4y - 2z &= 7 \\ 2z &= -7/2 \\ 0 &= 1/2 \end{aligned}$$

The fourth equation $0 = 1/2$ has no solutions, so the system also has no solutions.

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(c)
$$\begin{aligned} 2x - y - z + w &= 4 \\ x + y + 2z - w &= -1 \end{aligned}$$

▶ **Solution.** Apply Gauss's method to reduce the system to echelon form:

$$\xrightarrow{-(1/2)\rho_1+\rho_2} \begin{aligned} 2x - y - z + w &= 4 \\ (3/2)y + (5/2)z - (3/2)w &= -3 \end{aligned}$$

The leading variables are x and y, so the remaining variables z and w are the free variables. Now solve the second equation for y as a function of the free variables z and w and use back substitution to get x:

$$\begin{aligned} y &= -\frac{5}{3}z + w - 2 \\ x &= \frac{1}{2}y + \frac{1}{2}z - \frac{1}{2}w + 2 = \frac{1}{2}\left(-\frac{5}{3}z + w - 2\right) + \frac{1}{2}z - \frac{1}{2}w + 2 \\ &= -\frac{1}{3}z + 1. \end{aligned}$$

Thus, the solution set is

$$\left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -\frac{1}{3}z + 1 \\ -\frac{5}{3}z + w - 2 \\ z \\ w \end{bmatrix} \middle|\, z,\, w \in \mathbb{R} \right\}.$$

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(d)
$$\begin{aligned} x + y - 2z &= 0 \\ x - y \phantom{{}-2z} &= -3 \\ 3x - y - 2z &= 0 \end{aligned}$$

▶ **Solution.** For this system we will use the augmented matrix notation for doing the Gauss reduction to echelon form.

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & -1 & 0 & -3 \\ 3 & -1 & -2 & 0 \end{array}\right) \xrightarrow[-3\rho_1+\rho_3]{-\rho_1+\rho_2} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -2 & 2 & -3 \\ 0 & -4 & 4 & 0 \end{array}\right) \xrightarrow{-2\rho_2+\rho_3} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -2 & 2 & -3 \\ 0 & 0 & 0 & 6 \end{array}\right)$$

The third row represents the equation $0 = 6$, which has no solutions, so the solution set of the system is empty. ◀

2. For the second system in the first question, give the associated homogeneous system and find its solution set.

▶ **Solution.** The associated homogeneous system is

$$\begin{aligned} x + y + 2z &= 0 \\ x - 3y \phantom{{}+2z} &= 0 \\ 3x + y + 7z &= 0 \\ 2y + z &= 0 \end{aligned}$$

To solve this system, apply Gauss's method to reduce the system to echelon form:

$$\xrightarrow[-3\rho_1+\rho_3]{-\rho_1+\rho_2} \begin{aligned} x + y + 2z &= 0 \\ -4y - 2z &= 0 \\ -2y + z &= 0 \\ 2y + z &= 0 \end{aligned} \xrightarrow[(1/2)\rho_2+\rho_4]{-(1/2)\rho_2+\rho_3} \begin{aligned} x + y + 2z &= 0 \\ -4y - 2z &= 0 \\ 2z &= 0 \\ 0 &= 0 \end{aligned}$$

Solving by back substitution gives the unique solution $(x,\, y,\, z) = (0,\, 0,\, 0)$. ◀

3. Solve the linear system via Gauss's method.

(a) $$\begin{aligned} x+y-z&=3\\ 2x-y-z&=1\\ 3x+y+2z&=0 \end{aligned}$$

▶ **Solution.** Do Gauss reduction using the augmented matrix.

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 3\\ 2 & -1 & -1 & 1\\ 3 & 1 & 2 & 0 \end{array}\right) \xrightarrow[-3\rho_1+\rho_3]{-2\rho_1+\rho_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 3\\ 0 & -3 & 1 & -5\\ 0 & -2 & 5 & -9 \end{array}\right)$$
$$\xrightarrow{-(2/3)\rho_2+\rho_3} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 3\\ 0 & -3 & -1 & 1\\ 0 & 0 & 13/3 & -17/3 \end{array}\right)$$

This last matrix is the augmented matrix of the linear system in echelon form:

$$\begin{aligned} x+y-z&=3\\ -3y+z&=-5\\ 13/3z&=-17/3 \end{aligned}$$

This can be solved by back substitution: $z=-17/13$, $y=-(1/3)z+5/3=-(17/39)+5/3=16/13$ and $x=-y+z+3=-(16/13)-(17/13)+3=6/13$. Thus, the unique solution is

$$(x,\,y,\,z)=(6/13,\,16/13,\,-17/13).$$

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(b) $$\begin{aligned} x+y+2z&=0\\ 2x-y+z&=1\\ 4x+y+5z&=1 \end{aligned}$$

▶ **Solution.** Do Gauss reduction using the augmented matrix.

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0\\ 2 & -1 & 1 & 1\\ 4 & 1 & 5 & 1 \end{array}\right) \xrightarrow[-4\rho_1+\rho_3]{-2\rho_1+\rho_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0\\ 0 & -3 & -3 & 1\\ 0 & -3 & -3 & 1 \end{array}\right) \xrightarrow{-\rho_2+\rho_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0\\ 0 & -3 & -3 & 1\\ 0 & 0 & 0 & 0 \end{array}\right)$$

This last matrix is the augmented matrix of the linear system in echelon form:

$$\begin{aligned} x+y+2z&=0\\ -3y-3z&=1\\ 0&=0 \end{aligned}$$

Thus, x and y are the leading variables and z is a free variable. Solve for x and y by back substitution to get $y = -z - (1/3)$ and $x = -z + (1/3)$. Thus, the solution set is

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z + \frac{1}{3} \\ -z - \frac{1}{3} \\ z \end{bmatrix} \middle| \, z \in \mathbb{R} \right\}.$$

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