In all homework problems, it is not sufficient to show only the answers. *You must show your work.*

1. Solve the linear system via Gauss-Jordan reduction. Express your answers in the form of Theorem 3.1, Page 24 of the text.

(a)
$$\begin{aligned} 2x + y + 3z &= 1 \\ 4x + 3y + 5z &= 1 \\ 6x + 5y + 5z &= -3 \end{aligned}$$

▶ **Solution.** Use augmented matrices with Gauss-Jordan reduction:

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 4 & 3 & 5 & 1 \\ 6 & 5 & 5 & -3 \end{array}\right] \underset{-3\rho_1+\rho_3}{\xrightarrow{-2\rho_1+\rho_2}} \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & -4 & -6 \end{array}\right]$$

$$\xrightarrow[-2\rho_2+\rho_3]{} \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -2 & -4 \end{array}\right] \underset{-\frac{1}{2}\rho_3}{\xrightarrow{\frac{1}{2}\rho_1}} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\underset{-\frac{3}{2}\rho_3+\rho_1}{\xrightarrow{\rho_3+\rho_2}} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & -\frac{5}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{-\frac{1}{2}\rho_2+\rho_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

The last matrix is in reduced echelon form, so the solution can be read off as

$$(x,\, y,\, z) = (-3,\, 1,\, 2).$$

◀

(b)
$$\begin{aligned} x_1 + 3x_2 + x_3 + 5x_4 + x_5 &= 5 \\ x_2 + x_3 + 2x_4 + x_5 &= 4 \\ 2x_1 + 5x_2 \phantom{{}+x_3} + 7x_4 + x_5 &= 3 \end{aligned}$$

▶ **Solution.** Apply Gauss-Jordan reduction to the augmented matrix:

$$\left[\begin{array}{ccccc|c} 1 & 3 & 1 & 5 & 1 & 5 \\ 0 & 1 & 1 & 2 & 1 & 4 \\ 2 & 5 & 0 & 7 & 1 & 3 \end{array}\right] \xrightarrow{-2\rho_1+\rho_3} \left[\begin{array}{ccccc|c} 1 & 3 & 1 & 5 & 1 & 5 \\ 0 & 1 & 1 & 2 & 1 & 4 \\ 0 & -1 & -2 & -3 & -1 & -7 \end{array}\right]$$

$$\xrightarrow{\rho_2+\rho_3} \left[\begin{array}{ccccc|c} 1 & 3 & 1 & 5 & 1 & 5 \\ 0 & 1 & 1 & 2 & 1 & 4 \\ 0 & 0 & -1 & -1 & 0 & -3 \end{array}\right] \underset{\rho_3+\rho_1}{\xrightarrow{\rho_3+\rho_2}} \left[\begin{array}{ccccc|c} 1 & 3 & 0 & 4 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 & -3 \end{array}\right]$$

$$\underset{-\rho_3}{\xrightarrow{-3\rho_2+\rho_1}} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & -2 & -1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 3 \end{array}\right]$$

The last matrix is in reduced echelon form, which is the augmented matrix of the linear system in reduced echelon form

$$\begin{aligned} x_1 \qquad\quad + x_4 - 2x_5 &= -1 \\ x_2 \quad + x_4 + \ \ x_5 &= \ \ 1 \\ x_3 + x_4 \qquad\quad &= \ \ 3 \end{aligned}$$

Thus, the leading variables are x_1, x_2 and x_3, which means that the free variables are x_4 and x_5. Thus we can write the solutions as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1 - x_4 + 2x_5 \\ 1 - x_4 - x_5 \\ 3 - x_4 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_5$$

where x_4 and x_5 can be any real numbers. The term without the parameters x_4 and x_5 is the particular solution with $x_4 = x_5 = 0$, whereas the linear combination with the parameters x_4 and x_5 is the general solution of the associated homogeneous system. ◀

2. Consider the system of linear equations

$$\begin{aligned} x + y - \ \ z &= a \\ x - y + 2z &= b \end{aligned} \tag{1}$$

where a and b are some real numbers.

(a) Find the general solution of the associated homogeneous equation.

▶ **Solution.** The associated homogeneous system is

$$\begin{aligned} x + y - \ \ z &= 0 \\ x - y + 2z &= 0 \end{aligned}$$

Solve by Gauss-Jordan reduction:

$$\left[\begin{array}{rrr|r} 1 & 1 & -1 & 0 \\ 1 & -1 & 2 & 0 \end{array}\right] \xrightarrow{-\rho_1+\rho_2} \left[\begin{array}{rrr|r} 1 & 1 & -1 & 0 \\ 0 & -2 & 3 & 0 \end{array}\right]$$
$$\xrightarrow[-\frac{1}{2}\rho_2]{} \left[\begin{array}{rrr|r} 1 & 1 & -1 & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \end{array}\right] \xrightarrow{-\rho_2+\rho_1} \left[\begin{array}{rrr|r} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \end{array}\right]$$

The last matrix is in reduced echelon form, which is the augmented matrix of the following linear system in reduced echelon form

$$\begin{aligned} x \qquad + \tfrac{1}{2}z &= 0 \\ x_2 - \tfrac{3}{2}z &= 2 \end{aligned}$$

The leading variables are x and y, while the remaining variable z is the only free variable. Thus, the solution set is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1/2 \\ 3/2 \\ 1 \end{bmatrix} z$$

where z can be any real number. ◀

(b) A particular solution of Equation (1) when $a = 1$ and $b = 2$ is $(x, y, z) = (1, 1, 1)$. Find the general solution of Equation (1).

▶ **Solution.** The general solution in this case is the sum of the given particular solution and the general solution of the associated homogeneous equation. Thus, the general solution with $a = 1$, $b = 2$ is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1/2 \\ 3/2 \\ 1 \end{bmatrix} z$$

where z is an arbitrary real number. ◀

(c) Find a particular solution of Equation (1) when $a = -1$ and $b = -2$.

▶ **Solution.** Just multiply a particular solution for the case $a = 1$, $b = 2$ by -1. Thus, a particular solution is $(x, y, z) = (-1, -1, -1)$. ◀

(d) Find a particular solution of Equation (1) when $a = 3$ and $b = 6$.

▶ **Solution.** Just multiply a particular solution for the case $a = 1$, $b = 2$ by 3. Thus, a particular solution is $(x, y, z) = (3, 3, 3)$. ◀

[Remark: After you have done part (a), it is possible to immediately write down the solutions in the remaining parts. Of course, you need to explain how you got your answers.]

3. Consider the system of linear equations

$$\begin{aligned} x + x + kz &= 1 \\ x + ky + z &= 1 \\ kx + y + z &= 1 \end{aligned} \tag{2}$$

where k is some real number. For what value(s) of k does this system have

(a) a unique solution?

(b) no solution?

(c) infinitely many solutions?

(Justify your assertions.)

▶ **Solution.** Use augmented matrices with Gauss reduction to reduce to echelon form:

$$\left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 1 & k & 1 & 1 \\ k & 1 & 1 & 1 \end{array}\right] \xrightarrow[-k\rho_1+\rho_3]{-\rho_1+\rho_2} \left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ 0 & 1-k & 1-k^2 & 1-k \end{array}\right]$$

$$\xrightarrow[\rho_2+\rho_3]{} \left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ 0 & 0 & 2-k-k^2 & 1-k \end{array}\right]$$

The last matrix is in echelon form, so the system will have a unique solution if and only if all of the variables are leading variables. This will happen so long as $k - 1 \neq 0$ and $2 - k - k^2 \neq 0$. Since $2 - k - k^2 = (s + k)(1 - k) = 0$ if and only if $k = -2$ or $k = 1$, it follows that the system will have a unique solution provided $k \neq -2$ or 1.

If $k = -2$ then the augmented matrix is $\left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{array}\right]$. Since the last row represents the inconsistent equation $0 = 3$, there will be no solutions to the system if $k = -2$.

If $k = 1$ then the augmented matrix is $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 \end{array}\right]$. This matrix has no rows representing an inconsistent equation, so there are solutions. Moreover, there is only one leading variable, namely x, so there are free variables (namely, y and z). Thus, there are infinitely many solutions to the system in case $k = 1$. ◀