In all homework problems, it is not sufficient to show only the answers. *You must show your work.*

1. Determine whether each subset of $\mathcal{M}_{2\times 2}$ is a vector subspace by checking whether condition (2) of Lemma 9 Page 92 is satisfied for each subset.

   (a) V_1 is the collection of 2×2 matrices with 0 in the upper right entry. That is

   $$V_1 = \{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \mid a,\, c,\, d \in \mathbb{R} \}$$

   ▶ **Solution.** Let $\vec{v_1} = \begin{pmatrix} a_1 & 0 \\ c_1 & d_1 \end{pmatrix}$ and $\vec{v_2} = \begin{pmatrix} a_2 & 0 \\ c_2 & d_2 \end{pmatrix}$ be arbitrary elements of V_1 and let r and s be arbitrary real numbers. Then

   $$r\vec{v_1} + s\vec{v_2} = r \begin{pmatrix} a_1 & 0 \\ c_1 & d_1 \end{pmatrix} + s \begin{pmatrix} a_2 & 0 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} ra_1 + sa_2 & r\cdot 0 + s \cdot 0 \\ rc_1 + sc_2 & rd_1 + sd_2 \end{pmatrix} = \begin{pmatrix} \tilde{a} & 0 \\ \tilde{c} & \tilde{d} \end{pmatrix} \in V_1$$

   since the upper right hand entry is 0. Thus, V_1 satisfies condition (2) of Lemma 9, and hence V_1 is a subspace of $\mathcal{M}_{2\times 2}$. ◀

   (b)

   $$V_2 = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,\, b,\, c,\, d \in \mathbb{R} \text{ and } a + d = 1 \}$$

   ▶ **Solution.** $\vec{v_1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is in V_2 since for $\vec{v_1}$, $a + d = 1 + 0 = 1$. However, $2\vec{v_1} = 2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ is not in V_2 since for $2\vec{v_1}$, $a + d = 2 + 0 = 2 \neq 1$. Thus, V_2 fails to be closed under scalar multiplication, and hence is not a subspace. You could also check that V_2 is not closed under vector addition, which would also show that it is not a subspace. ◀

2. Determine (with justification) if each set is linearly independent (in the natural vector space).

   (a) $\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \}$

   ▶ **Solution.** Suppose there is a linear dependence relation

   $$c_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Adding the left hand side together gives a system of linear equations for c_1, c_2:

$$\begin{aligned} c_1 - c_2 &= 0 \\ 2c_1 + c_2 &= 0 \\ 0 &= 0 \end{aligned}$$

The only solution of this system is $c_1 = c_2 = 0$, so the set of two vectors is linearly independent. ◀

(b) $\{(1\ \ 3\ \ 1), (-1\ \ 4\ \ 3), (-1\ \ 11\ \ 7)\}$

▶ **Solution.** Suppose there is a linear dependence relation

$$c_1(1\ \ 3\ \ 1) + c_2(-1\ \ 4\ \ 3) + c_3(-1\ \ 11\ \ 7) = (0\ \ 0\ \ 0).$$

Adding the left hand side together gives a system of linear equations for c_1, c_2, c_3:

$$\begin{aligned} c_1 - c_2 - c_3 &= 0 \\ 3c_1 + 4c_2 + 11c_3 &= 0 \\ c_1 + 3c_2 + 7c_3 &= 0 \end{aligned}$$

The augmented matrix of this system is $\left[\begin{array}{rrr|r} 1 & -1 & -1 & 0 \\ 3 & 4 & 11 & 0 \\ 1 & 3 & 7 & 0 \end{array}\right]$, which after Gauss-Jordan reduction, produces the reduced echelon matrix $\left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. This gives the equivalent system of linear equations

$$\begin{aligned} c_1 + c_3 &= 0 \\ c_2 + 2c_3 &= 0 \\ 0 &= 0 \end{aligned}$$

Thus, taking, for example, $c_3 = 1$ gives a nontrivial linear dependence relation

$$(-1)(1\ \ 3\ \ 1) + (-2)(-1\ \ 4\ \ 3) + (1)(-1\ \ 11\ \ 7) = (0\ \ 0\ \ 0).$$

Thus, this set of vectors is linearly dependent. ◀

(c) $\left\{\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 4 \end{pmatrix}\right\}$

▶ **Solution.** This set is linearly dependent since it contains the zero vector $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ of the vector space of 2×2 matrices. ◀

3. Determine if the given vector, is in the span of the given set, inside the given vector space.

(a) $\begin{pmatrix}1\\0\\3\end{pmatrix}$, $\left\{\begin{pmatrix}2\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}\right\}$, in $\mathbb{R}^3$

▶ **Solution.** Try to write:

$$c_1\begin{pmatrix}2\\1\\-1\end{pmatrix} + c_2\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\3\end{pmatrix}.$$

If this is possible, then c_1 and c_2 must satisfy the system of linear equations

$$\begin{aligned} 2c_1 + c_2 &= 1 \\ c_1 - c_2 &= 0. \\ -c_1 + c_2 &= 3 \end{aligned}$$

Adding the second and third equations gives the inconsistent equation $0 = 3$. This means that there is no choice of c_1 and c_2 that satisfies these equations, so $\begin{pmatrix}1\\0\\3\end{pmatrix}$ is not in the span of $\left\{\begin{pmatrix}2\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}\right\}$. ◀

(b) $x^2 - 4x^3$, $\{x^2,\ 2x + x^2,\ x + x^3\}$, in $\mathcal{P}_3$.

▶ **Solution.** Try to write: $x^2 - 4x^3 = c_1x^2 + c_2(2x + x^2) + c_3(x + x^3)$. By comparing coefficients of x, x^2 and x^3 on the left and right we obtain a system of linear equations

$$\begin{aligned} 2c_2 + c_3 &= 0 \\ c_1 + c_2 \quad &= 1 \\ c_3 &= -4 \end{aligned}$$

that must be satisfied. Solving this system (for example, by Gauss-Jordan reduction) gives a unique solution $c_1 = -1$, $c_2 = 2$, $c_3 = -4$. Thus, $x^2 - 4x^3$ is in the span of $\{x^2,\ 2x + x^2,\ x + x^3\}$. In particular,

$$x^2 - 4x^3 = (-1)x^2 + 2(2x + x^2) - 4(x + x^3).$$

◀

4. Determine which of these sets spans $\mathbb{R}^3$. That is, which of the sets has the property that any vector in $\mathbb{R}^3$ can be expressed as a linear combination of the elements of the set?

(a) $\left\{\begin{pmatrix}1\\1\\-2\end{pmatrix}, \begin{pmatrix}1\\-2\\1\end{pmatrix}, \begin{pmatrix}-2\\1\\1\end{pmatrix}\right\}$

▶ **Solution.** We need to determine if every vector in $\mathbb{R}^3$ can be written as a linear combination of the three vectors in the set. That is, for an arbitrary vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, try to find scalars c_1, c_2, c_3 such that

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$$

Comparing coefficients of each component of the vectors on the left and right, we obtain a system of linear equations

$$\begin{aligned} c_1 + \ \ c_2 - 2c_3 &= a \\ c_1 - 2c_2 + \ \ c_3 &= b \\ -2c_1 + \ \ c_2 + \ \ c_3 &= c \end{aligned}$$

that must be solvable for all choices of a, b, and c. Try to solve this system by applying Gauss-Jordan reduction to the augmented matrix.

$$\left[\begin{array}{rrr|c} 1 & 1 & -2 & a \\ 1 & -2 & 1 & b \\ -2 & 1 & 1 & c \end{array}\right] \xrightarrow[2\rho_1+\rho_3]{-\rho_1+\rho_2} \left[\begin{array}{rrr|c} 1 & 1 & -2 & a \\ 0 & -3 & 3 & b-a \\ 0 & 3 & -3 & c+2a \end{array}\right]$$

$$\xrightarrow[\rho_2+\rho_3]{} \left[\begin{array}{rrr|c} 1 & 1 & -2 & a \\ 0 & -3 & 3 & b-a \\ 0 & 0 & 0 & b+c-a \end{array}\right]$$

The third row corresponds to the equation $0 = b+c-a$, so the system is solvable only if $b+c-a = 0$. Since this is not true for all a, b, c, in particular, for example, for $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, it follows that $\mathbb{R}^3$ is not the span of the given set. ◀

(b) $\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \right\}$

▶ **Solution.** For an arbitrary vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, try to find scalars c_1, c_2, c_3, and c_4 such that

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} + c_4 \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}.$$

Comparing coefficients of each component of the vectors on the left and right, we obtain a system of linear equations

$$\begin{aligned} c_1 + 4c_2 - 2c_3 + 2c_4 &= a \\ 2c_2 \qquad + \ \ c_4 &= b \\ -c_1 + \qquad\qquad + 3c_4 &= c \end{aligned}$$

that must be solvable for all choices of a, b, and c. Try to solve this system by applying Gauss-Jordan reduction to the augmented matrix.

$$\left[\begin{array}{cccc|c} 1 & 4 & -2 & 2 & a \\ 0 & 1 & 0 & 1 & b \\ -1 & 0 & 0 & 3 & c \end{array}\right] \xrightarrow[\rho_1+\rho_3]{} \left[\begin{array}{cccc|c} 1 & 4 & -2 & 2 & a \\ 0 & 1 & 0 & 1 & b \\ 0 & 4 & -2 & 5 & a+c \end{array}\right]$$

$$\xrightarrow[-4\rho_2+\rho_3]{} \left[\begin{array}{cccc|c} 1 & 4 & -2 & 2 & a \\ 0 & 1 & 0 & 1 & b \\ 0 & 0 & -2 & 1 & a-4b+c \end{array}\right]$$

This last matrix is in echelon form, with every row having a leading variable, so the original system is solvable for all choices of the last column. Hence, $\mathbb{R}^3$ is the span of this set. In fact, the echelon matrix actually shows that the first 3 vectors in the set already span $\mathbb{R}^3$. ◀