In all homework problems, it is not sufficient to show only the answers. *You must show your work.* These exercises are based on Chapter Two.III.1-Two.III.2 from the text.

1. Find a basis for each vector space. Verify that it is a basis.

   (a) The subspace $M = \{\, a + bx + cx^2 + dx^3 \mid a - 2b + c - d = 0 \,\}$ of $\mathcal{P}_3$.

   ▶ **Solution.** Identify the vector $a + bx + cx^2 + cx^3 \in \mathcal{P}_3$ with its vector of coefficients $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ in $\mathbb{R}^4$. Solve the homogeneous defining condition $a - 2b + c - d = 0$ as $a = 2b - c + d$, $b = b$, $c = c$, and $d = d$ in $\mathbb{R}^4$ to get

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2b - c + d \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} b + \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} c + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} d,$$

   which identifies the homogeneous solution set as the span of the set

$$\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \}.$$

   Since $\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ corresponds to $2 + x$, $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ corresponds to $-1 + x^2$, and $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ corresponds to $1 + x^3$, this gives a description of M as the span of a set of three vectors:

$$M = \{\, (2 + x) \cdot b + (-1 + x^2) \cdot c + (1 + x^3) \cdot d \mid b, c, d \in \mathbb{R} \,\}$$

   To show that this three-vector set is a basis, what remains is for us to verify that it is linearly independent.

$$0 + 0x + 0x^2 + 0x^3 = (2 + x) \cdot c_1 + (-1 + x^2) \cdot c_2 + (1 + x^3) \cdot c_3$$

   From the x terms we see that $c_1 = 0$. From the x^2 terms we see that $c_2 = 0$. The x^3 terms give that $c_3 = 0$. ◀

   (b) The subspace W of $\mathcal{M}_{2\times 2}$:

$$W = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a - c = 0 \}$$

▶ **Solution.** First parametrize the description (note that the fact that b and d are not mentioned in the description of W does not mean they are zero or absent, it means that they are unrestricted).

$$W = \{\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\cdot b + \begin{pmatrix}1 & 0\\1 & 0\end{pmatrix}\cdot c + \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\cdot d \mid b, c, d \in \mathbb{R}\}$$

That gives W as the span of a three element set. We will be done if we show that the set is linearly independent. So consider a dependence relation

$$\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\cdot c_1 + \begin{pmatrix}1 & 0\\1 & 0\end{pmatrix}\cdot c_2 + \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\cdot c_3$$

Using the upper right entries we see that $c_1 = 0$. The upper left entries give that $c_2 = 0$, and the lower left entries show that $c_3 = 0$. ◀

(c) The subspace V of $\mathbb{R}^4$:

$$V = \{\begin{pmatrix}x\\y\\z\\w\end{pmatrix} \in \mathbb{R}^4 \mid x - w + z = 0\}$$

▶ **Solution.** Parametrize to get this description of the space.

$$\{\begin{pmatrix}w-z\\y\\z\\w\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}y + \begin{pmatrix}-1\\0\\1\\0\end{pmatrix}z + \begin{pmatrix}1\\0\\0\\1\end{pmatrix}w \mid y, z, w \in \mathbb{R}\}$$

That gives the space as the span of the three-vector set. To show the three vector set makes a basis we check that it is linearly independent.

$$\begin{pmatrix}0\\0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}c_1 + \begin{pmatrix}-1\\0\\1\\0\end{pmatrix}c_2 + \begin{pmatrix}1\\0\\0\\1\end{pmatrix}c_3$$

The second components give that $c_1 = 0$, and the third and fourth components give that $c_2 = 0$ and $c_3 = 0$. So one basis is this.

$$\langle \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix} \rangle$$

The dimension is the number of vectors in a basis: 3. ◀

(d) The subspace $N = \{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_0 + a_1 = 0 \text{ and } a_2 - 2a_3 = 0\}$ of $\mathcal{P}_3$

▶ **Solution.** The restrictions form a two-equations, four-unknowns linear system. Parametrizing that system to express the leading variables in terms of those that are free gives $a_0 = -a_1$, $a_1 = a_1$, $a_2 = 2a_3$, and $a_3 = a_3$.

$$\{-a_1 + a_1x + 2a_3x^2 + a_3x^3 \mid a_1, a_3 \in \mathbb{R}\} = \{(-1+x)\cdot a_1 + (2x^2 + x^3)\cdot a_3 \mid a_1, a_3 \in \mathbb{R}\}$$

That description shows that the space is the span of the two-element set $\{-1+x, x^2+x^3\}$. We will be done if we show the set is linearly independent. This relationship

$$0 + 0x + 0x^2 + 0x^3 = (-1+x)\cdot c_1 + (2x^2 + x^3)\cdot c_2$$

gives that $c_1 = 0$ from the constant terms, and $c_2 = 0$ from the cubic terms. One basis for the space is $\langle -1+x, 2x^2+x^3\rangle$. This is a two-dimensional space. ◀

2. Give two different bases for $\mathbb{R}^3$. Verify that each is a basis.

▶ **Solution.** Obviously there are many different correct choices of bases. The natural basis for $\mathbb{R}^3$ is this.

$$\mathcal{E}_3 = \langle \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \rangle$$

The verification that it spans $\mathbb{R}^3$ is easy: for any $x, y, z \in \mathbb{R}$ this equation has a solution,

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}\cdot c_1 + \begin{pmatrix}0\\1\\0\end{pmatrix}\cdot c_2 + \begin{pmatrix}0\\0\\1\end{pmatrix}\cdot c_3 \qquad (*)$$

namely, $c_1 = x$, $c_2 = y$, and $c_3 = z$. Further, the set is linearly independent since the relationship

$$\begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}\cdot c_1 + \begin{pmatrix}0\\1\\0\end{pmatrix}\cdot c_2 + \begin{pmatrix}0\\0\\1\end{pmatrix}\cdot c_3$$

obviously has only the trivial solution. (*Comment.* We could have done the argument in one step by observing that equation $(*)$ shows that each vector from $\mathbb{R}^3$ is represented with respect to this basis $\mathcal{E}_3$ in one and only one way.)

This is a second basis for $\mathbb{R}^3$.

$$B = \langle \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \rangle$$

To verify that that it spans $\mathbb{R}^3$ suppose $x, y, z \in \mathbb{R}$, then

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot c_1 + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot c_2 + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot c_3$$

has a solution, namely $c_1 = x - y$, $c_2 = y - z$, and $c_3 = z$. Further, the set B is linearly independent since in the relationship

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot c_1 + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot c_2 + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot c_3$$

the third components give that $c_3 = 0$, then the second components give that $c_2 = 0$, and with those two the first components give that $c_1 = 0$. ◀

3. Represent the vector with respect to each of the two bases.

$$\vec{v} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \quad B_1 = \langle \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle, \; B_2 = \langle \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \rangle$$

▶ **Solution.** Writing

$$\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot c_1 + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot c_2$$

gives a linear system for c_1, c_2:

$$\begin{aligned} c_1 + c_2 &= 3 \\ -c_1 + c_2 &= -1 \end{aligned}$$

Solving this system gives $c_1 = 2$ and $c_2 = 1$, so that

$$\operatorname{Rep}_{B_1}(\begin{pmatrix} 3 \\ -1 \end{pmatrix}) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}_{B_1}$$

Similarly, writing

$$\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot c_1 + \begin{pmatrix} 1 \\ 3 \end{pmatrix} \cdot c_2$$

gives a linear system for c_1, c_2:

$$\begin{aligned} c_1 + c_2 &= 3 \\ 2c_1 + 3c_2 &= -1 \end{aligned}$$

Solving this system gives $c_1 = 10$ and $c_2 = -7$, so that

$$\operatorname{Rep}_{B_2}(\begin{pmatrix} 3 \\ -1 \end{pmatrix}) = \begin{pmatrix} 10 \\ -7 \end{pmatrix}_{B_2}$$

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