In all homework problems, it is not sufficient to show only the answers. *You must show your work.* These exercises are based on Chapter Two.III.3 from the text.

1. Let $A = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 1 \\ -1 & 2 & 7 \end{bmatrix}$.

   (a) Find a basis for the row space of the matrix A.

   (b) Find a basis for the column space of the matrix A.

   (c) Find a basis for the null space of the matrix A. (Recall that the null space of A is the solution space of the homogeneous linear system $A\vec{x} = \vec{0}$.)

   (d) Determine if each of the vectors $\vec{v} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 2 & 1 & 1 \end{bmatrix}$ is in the row space of A.

   (e) Determine if each of the vectors $\vec{a} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ is in the column space of A.

▶ **Solution.** First find the reduced row-echelon form of the matrix A using Gauss-Jordan reduction:

$$\begin{aligned}
\begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 1 \\ -1 & 2 & 7 \end{bmatrix} &\xrightarrow{\rho_1 \leftrightarrow \rho_2} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 3 \\ -1 & 2 & 7 \end{bmatrix} \\
\xrightarrow{-\rho_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ -1 & 2 & 7 \end{bmatrix} &\xrightarrow[\rho_1+\rho_3]{} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{bmatrix} \\
\xrightarrow[-2\rho_2+\rho_3]{} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} &
\end{aligned}$$

The last matrix is rref(A), the reduced row-echelon form of A. From this we can read off the answers to the first three parts:

(a) The nonzero rows of rref(A) form a basis of the row space of A. Thus, a basis is

$$\mathcal{R} = \langle \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 3 \end{bmatrix} \rangle.$$

(b) The first two columns of rref(A) contain the leading 1's of the nonzero rows of rref(A). The corresponding columns in A form a basis of the column space of A. Thus, a basis is

$$\mathcal{C} = \langle \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \rangle.$$

(c) We can read off the solutions to $A\vec{x} = \vec{0}$ from rref(A) (leading variables are x_1, x_2 and the only free variable is x_3) to get the nullspace of A is

$$\text{Null Space}(A) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -3x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} x_3 \mid x_3 \text{ is arbitrary} \right\}$$

Thus a basis of the null space of A is the single vector $\begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$.

(d) For $\vec{v}$ to be in the row space of A we must have

$$\vec{v} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 & 3 \end{bmatrix}.$$

This gives the equations $c_1 = 1$, $c_2 = 1$ and $-c_1 + 3c_2 = 1$. This is an inconsistent system of equations (the third equation is not possible if the first two are satisfied). Thus $\vec{v}$ is not in the row space of A.

However, $\vec{w} = \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 3 \end{bmatrix}$, so $\vec{w}$ is in the row space of A.

(e) In part (b), a basis $\mathcal{C}$ for the column space of A was found. To see is $\vec{a}$ is in the column space of A, try to write $\vec{a}$ as a linear combination of this basis:

$$\vec{a} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} c_2 \\ -c_1 \\ -c_1 + 2c_2 \end{bmatrix}$$

This gives the equations $c_2 = 1$, $-c_1 = 1$, $-c_1 + 2c_2 = 3$. These are satisfied for $c_1 = -1$, $c_2 = 1$ so

$$\vec{a} = -\begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

so $\vec{a}$ is in the column space of A. However, the same calculation with $\vec{b}$ would lead to the equations $c_2 = 3$, $-c_1 = 1$, $-c_1 + 2c_2 = 1$. These equations are inconsistent, so $\vec{b}$ is not in the column space of A.

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2. In each part (a)–(b) assume that the matrix A is row equivalent to the matrix B. Without additional calculations, list rank(A) and dim(Nullspace(A)). Then find bases for Colspace(A), Rowspace(A), and Nullspace(A).

(a) $A = \begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 2 & 6 & 6 & 0 & -3 \\ 3 & 9 & 3 & 6 & -3 \\ 3 & 9 & 0 & 9 & 0 \end{bmatrix}$, $\quad B = \begin{bmatrix} 1 & 3 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

▶ **Solution.**
- There are 3 nonzero rows in $B = \mathrm{rref}(A)$, so $\mathrm{rank}(A) = 3$.
- There are 2 columns of B not containing a leading 1, so $\dim(\mathrm{Nullspace}(A)) = 2$.
- A basis for $\mathrm{Colspace}(A)$ consists of the columns of A corresponding to the columns of B containing a leading 1. That is, columns 1, 3, and 5 of A. Thus, a basis is

$$\mathcal{C} = \left\langle \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -3 \\ 0 \end{bmatrix} \right\rangle.$$

- A basis of the row space of A consists of the nonzero rows of the row equivalent echelon matrix B. Thus, a basis is

$$\mathcal{B} = \langle \begin{bmatrix} 1 & 3 & 0 & 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix} \rangle.$$

- The null space of A is the same as the null space of B, so solve the homogeneous system $B\vec{x} = \vec{0}$:

$$\begin{aligned} c_1 + 3c_2 \qquad\quad + 3c_4 \quad &= 0 \\ c_3 - c_4 \qquad\quad &= 0 \\ c_5 &= 0 \\ 0 &= 0 \end{aligned}$$

The free variables are c_2 and c_4 so the null space of A is thus

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} -3c_2 - 3c_4 \\ c_2 \\ c_4 \\ c_4 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} c_2 + \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} c_4$$

where c_2 and c_4 are arbitrary. Thus a basis of the null space of A is

$$\mathcal{N} = \left\langle \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\rangle.$$

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(b) $A = \begin{bmatrix} 2 & 6 & -6 & 6 & 3 & 6 \\ -2 & -3 & 6 & -3 & 0 & -6 \\ 4 & 9 & -12 & 9 & 3 & 12 \\ -2 & 3 & 6 & 3 & 3 & -6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

▶ **Solution.** There was a misprint in the matrix B. The lower right entry should have been 0, rather than 1. With a 1 in that position the matrix B is **not** row equivalent to A, nor is it is reduced row echelon form. The solution given is based on correcting this misprint.

- There are 3 nonzero rows in $B = \operatorname{rref}(A)$, so $\operatorname{rank}(A) = 3$.
- There are 3 columns of B not containing a leading 1, so $\dim(\operatorname{Nullspace}(A)) = 3$.
- A basis for $\operatorname{Colspace}(A)$ consists of the columns of A corresponding to the columns of B containing a leading 1. That is, columns 1, 2, and 5 of A. Thus, a basis is

$$\mathcal{C} = \left\langle \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \\ 3 \end{bmatrix} \right\rangle.$$

- A basis of the row space of A consists of the nonzero rows of the row equivalent echelon matrix B. Thus, a basis is

$$\mathcal{B} = \langle \begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \rangle.$$

- The null space of A is the same as the null space of B, so solve the homogeneous system $B\vec{x} = \vec{0}$:

$$\begin{array}{rcl} c_1 \quad -3c_3 \qquad\qquad + 3c_6 &=& 0 \\ c_2 \qquad + c_4 \qquad\qquad &=& 0 \\ c_5 \qquad &=& 0 \\ 0 &=& 0 \end{array}$$

The free variables are c_2, c_4, and c_6 so the null space of A is thus

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{bmatrix} = \begin{bmatrix} 3c_3 - 3c_6 \\ -c_4 \\ c_3 \\ c_4 \\ 0 \\ c_6 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} c_3 + \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} c_4 + \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} c_6$$

where c_3, c_4, and c_6 are arbitrary. Thus a basis of the null space of A is

$$\mathcal{N} = \left\langle \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle.$$

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3. Answer each of the following questions related to the rank of an $m \times n$ matrix A.

   (a) If a 4×7 matrix A has rank 3, find the dimension of Nulllspace(A) and Rowspace(A).

   (b) If the null space of an 8×7 matrix A is 5-dimensional, what is the dimension of the column space of A?

   (c) If the null space of an 8×5 matrix A is 3-dimensional, what is the dimension of the row space of A?

   (d) If A is a 7×5 matrix, what is the largest possible rank of A?

   (e) If A is a 5×7 matrix, what is the largest possible rank of A?

▶ **Solution.** All of the answers are based on the rank-nullity theorem proved in class. Recall that this theorem states:

Theorem. *If A is an $m \times n$ matrix, then* $\operatorname{rank}(A) + \dim(\textit{Nullspace}(A)) = n$*, where n is the number of columns of A.*

Also recall that $\operatorname{rank}(A) = \dim(\text{Rowspace})(A) = \dim(\text{Colspace}(A))$. Thus the answers to the questions are:

(a) $\dim(\text{Rowspace}(A)) = \operatorname{rank}(A) = 3$, so $\dim(\text{Nullspace}(A)) = 7 - \operatorname{rank}(A) = 4$.

(b) If $\dim(\text{Nullspace}(A)) = 5$, then $\operatorname{rank}(A) = 7 - \dim(\text{Nullspace}(A)) = 2$. Thus, $\dim(\text{Colspace}(A)) = \operatorname{rank}(A) = 2$.

(c) If $\dim(\text{Nullspace}(A)) = 3$, then $\operatorname{rank}(A) = 5 - \dim(\text{Nullspace}(A)) = 2$. Thus, $\dim(\text{Rowspace}(A)) = \operatorname{rank}(A) = 2$.

(d) Since $\dim(\text{Rowspace}(A)) \leq$ the number of rows of A and $\dim(\text{Colspace}(A)) \leq$ the number of columns of A, and each of these dimensions is equal to the $\operatorname{rank}(A)$, it follows that if A is $m \times n$, then $\operatorname{rank}(A) \leq \min(m, n)$. Thus, the largest possible rank for a 7×5 matrix is 5.

(e) The largest possible rank for a 5×7 matrix is also 5.

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