In all homework problems, it is not sufficient to show only the answers. *You must show your work.* These exercises are based on Chapter Three.I and Three.II from the text.

1. Let $f : \mathcal{P}_1 \to \mathbb{R}^2$ be defined by

$$f(a+bx) = \begin{bmatrix} a+b \\ a-b \end{bmatrix}.$$

(a) Find the image of each of the following elements of the domain:
(i) $1-x$ (ii) $-1+3x$ (iii) $4+4x$

(b) For each of the following vectors $\begin{bmatrix} c \\ d \end{bmatrix} \in \mathbb{R}^2$, find a vector $a+bx \in \mathcal{P}_1$ with $f(a+bx) = \begin{bmatrix} c \\ d \end{bmatrix}$: (i) $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$ (iii) $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$

(c) Show that f is an isomorphism.

▶ **Solution.** (a) (i) $f(1-x) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, (ii) $f(-1+3x) = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$, (iii) $f(4+4x) = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$

(b) (i) $f(3-x) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, (ii) $f(\frac{5}{2}+\frac{5}{2}x) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$, (iii) $f(3+x) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

(c) Let $\vec{v}_1 = a_1 + b_1 x$ and $\vec{v}_2 = a_2 + b_2 x$ be arbitrary vectors in $\mathcal{P}_1$ and let $c_1, c_2 \in \mathbb{R}$ be arbitrary. Then

$$\begin{aligned} f(c_1\vec{v}_1 + c_2\vec{v}_2) &= f(c_1(a_1+b_1x) + c_2(a_2+b_2x)) \\ &= f((c_1a_1+c_2a_2) + (c_1b_1+c_2b_2)x) \\ &= \begin{bmatrix} (c_1a_1+c_2a_2) + (c_1b_1+c_2b_2) \\ (c_1a_1+c_2a_2) - (c_1b_1+c_2b_2) \end{bmatrix} \\ &= \begin{bmatrix} c_1(a_1+b_1) + c_2(a_2+b_2) \\ c_1(a_1-b_1) + c_2(a_2-b_2) \end{bmatrix} \\ &= c_1 \begin{bmatrix} a_1+b_1 \\ a_1-b_1 \end{bmatrix} + c_2 \begin{bmatrix} a_2+b_2 \\ a_2-b_2 \end{bmatrix} \\ &= c_1 f(\vec{v}_1) + c_2 f(\vec{v}_2). \end{aligned}$$

Hence f is a homomorphism.
To show it is an isomorphism, it is also necessary to show that f is one-to-one and onto. Given $\begin{bmatrix} c \\ d \end{bmatrix} \in \mathbb{R}^2$, $f(\frac{c+d}{2} + \frac{c-d}{2}x) = \begin{bmatrix} c \\ d \end{bmatrix} \in \mathbb{R}^2$ so f is onto. To show that f is one-to-one, suppose $f(a_1+b_1x) = f(a_2+b_2x)$. Then

$$\begin{bmatrix} a_1+b_1 \\ a_1-b_1 \end{bmatrix} = \begin{bmatrix} a_2+b_2 \\ a_2-b_2 \end{bmatrix}.$$

This means that $a_1+b_1 = a_2+b_2$ and $a_1-b_1 = a_2 = b_2$. Adding these two equations gives $2a_1 = 2a_2$, so $a_1 = a_2$ and subtracting the equations shows $b_1 = b_2$. Thus, $a_1 + b_1x = a_2 + b_2x$ and f is one-to-one. Therefore, f is an isomorphism.

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2. Neither of the following functions $f : \mathbb{R}^2 \to \mathbb{R}^2$ is an isomorphism. For each function identify a property in the definition of isomorphism that fails, and verify that that property fails.

   (a) $f(\begin{bmatrix} a \\ b \end{bmatrix}) = \begin{bmatrix} a+b \\ ab \end{bmatrix}$. (b) $f(\begin{bmatrix} a \\ b \end{bmatrix}) = \begin{bmatrix} a-b \\ 2a-2b \end{bmatrix}$.

▶ **Solution.** (a) $f(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $f(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ but $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and

$$f(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = f(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) + f(\begin{bmatrix} 0 \\ 1 \end{bmatrix}).$$

Therefore, f does not preserve vector addition and hence is not an isomorphism (or even a homomorphism). It is also true that f is not one-to-one since $f(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $f(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

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3. For which n is the space isomorphic to $\mathbb{R}^n$?

   (a) $\mathcal{P}_3$ (b) $\mathcal{M}_{2\times 3}$ (c) The null space of $A = \begin{bmatrix} 1 & 2 & 0 & -1 & 4 \\ 0 & 0 & 1 & 3 & -2 \end{bmatrix}$

▶ **Solution.** (a) $n = 4$, (b) $n = 6$, (c) $n = 3$ since there are 3 free variables for the linear system $A\vec{x} = \vec{0}$. ◀

4. Verify that each map is a homomorphism.

   (a) $f : \mathbb{R}^2 \to \mathcal{P}_2$ given by $f(\begin{bmatrix} a \\ b \end{bmatrix}) = (a+b)x + (a-b)x^2$.

   (b) $g : \mathbb{R}^2 \to \mathbb{R}^3$ given by $f(\begin{bmatrix} a \\ b \end{bmatrix}) = \begin{bmatrix} a-b \\ 0 \\ a+b \end{bmatrix}$.

▶ **Solution.** (a) Let $\vec{v}_1 = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$ be arbitrary vectors in $\mathbb{R}^2$ and let c_1, $c_2 \in \mathbb{R}$ be arbitrary. Then

$$\begin{aligned}
f(c_1\vec{v}_1 + c_2\vec{v}_2) &= f(c_1 \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + c_2 \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}) \\
&= f(\begin{bmatrix} c_1a_1 + c_2a_2 \\ c_1b_1 + c_2b_2 \end{bmatrix}) \\
&= ((c_1a_1 + c_2a_2) + (c_1b_1 + c_2b_2)) + ((c_1a_1 + c_2a_2) - (c_1b_1 + c_2b_2))x \\
&= c_1((a_1 + b_1) + (a_1 - b_1)x) + c_2((a_2 + b_2) + (a_2 - b_2)x) \\
&= c_1 f(\vec{v}_1) + c_2 f(\vec{v}_2).
\end{aligned}$$

Hence f is a homomorphism.

(b) Let $\vec{v}_1 = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$ be arbitrary vectors in $\mathbb{R}^2$ and let c_1, $c_2 \in \mathbb{R}$ be arbitrary. Then

$$\begin{aligned}
f(c_1\vec{v}_1 + c_2\vec{v}_2) &= f(c_1 \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + c_2 \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}) \\
&= f(\begin{bmatrix} c_1a_1 + c_2a_2 \\ c_1b_1 + c_2b_2 \end{bmatrix}) \\
&= \begin{bmatrix} (c_1a_1 + c_2a_2) - (c_1b_1 + c_2b_2) \\ 0 \\ (c_1a_1 + c_2a_2) + (c_1b_1 + c_2b_2) \end{bmatrix} \\
&= \begin{bmatrix} (c_1(a_1 - b_1) + c_2(a_2 - b_2) \\ 0 \\ (c_1(a_1 + b_1) + c_2(a_2 + b_2) \end{bmatrix} \\
&= c_1 \begin{bmatrix} a_1 - b_1 \\ 0 \\ a_1 + b_1 \end{bmatrix} + c_2 \begin{bmatrix} a_2 - b_2 \\ 0 \\ a_2 + b_2 \end{bmatrix} \\
&= c_1 f(\vec{v}_1) + c_2 f(\vec{v}_2).
\end{aligned}$$

Hence f is a homomorphism.

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5. Find the (i) range space, (ii) rank, (iii) null space, and (iv) nullity for each of the following homomorphisms.

   (a) $f : \mathbb{R}^2 \to \mathcal{P}_3$ given by $f(\begin{bmatrix} a \\ b \end{bmatrix}) = (a + b) + (a + b)x + (a + b)x^2$.

   (b) $g : \mathcal{M}_{2\times 2} \to \mathbb{R}$ given by $f(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = b - c$.

▶ **Solution.** (a) Range $(f) = \text{Span}\langle 1 + x + x^2\rangle$ since $f(\begin{bmatrix} a \\ b \end{bmatrix}) = (a+b) + (a+b)x + (a+b)x^2 = (a+b)(1+x+x^2)$, where a and b are arbitrary. Thus, $\text{rank}(f) = 1$. $\begin{bmatrix} a \\ b \end{bmatrix}$ is in the null space of f if and only if $a + b = 0$. Thus, the null space of A is $\text{Span}\langle \begin{bmatrix} -1 \\ 1 \end{bmatrix}\rangle$. Hence the nullity of f is 1.

(b) The range of g is $\mathbb{R}$, so the rank of f is 1. The null space of g is all 2×2 matrices with $b = c$. Thus, the null space is

$$\{\begin{bmatrix} a & b \\ b & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$$

A basis for this subspace is

$$\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\rangle.$$

Thus, the nullity of g is 3.

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