In all homework problems, it is not sufficient to show only the answers. *You must show your work.* These exercises are based on Chapter Three.I and Three.II from the text.

1. Assume that each matrix represents a map $h\colon \mathbb{R}^n \to \mathbb{R}^m$ with respect to the standard bases. In each case, (i) state m and n (ii) find $\mathscr{R}(h)$ (range space of h) and rank(h) (iii) find $\mathscr{N}(h)$ (null space of h) and nullity(h), and (iv) state whether the map is onto and whether it is one-to-one.

   (a) $\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 3 \\ 2 & 3 & 4 \\ -2 & -1 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$

▶ **Solution.** (a) For parts of the answer we will need to solve this system

$$\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \qquad (*)$$

for x and y so we do that calculation first.

$$\left[\begin{array}{cc|c} 2 & 1 & a \\ -1 & 3 & b \end{array}\right] \xrightarrow{(1/2)\rho_1+\rho_2} \xrightarrow[(2/7)\rho_2]{(1/2)\rho_1} \xrightarrow{-(1/2)\rho_2+\rho_1} \left[\begin{array}{cc|c} 1 & 0 & (3/7)a-(1/7)b \\ 0 & 1 & (1/7)a+(2/7)b \end{array}\right]$$

(i) The dimension of the domain space is the number of columns $m = 2$, and the dimension of the codomain space is the number of rows $m = 2$.

(ii) For all

$$\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$$

in equation $(*)$ the system has a solution, by the calculation. So the range space is all of the codomain $\mathscr{R}(h) = \mathbb{R}^2$. The map's rank is the dimension of the range, 2

(iii) Again by the calculation, setting $a = b = 0$ in equation $(*)$ gives that $x = y = 0$. The null space is the trivial subspace of the domain.

$$\mathscr{N}(h) = \{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\}$$

The nullity is the dimension of that null space, 0.

(iv) The map is onto because the range space is all of the codomain. The map is one-to-one because the null space is trivial.

(b) We will need to solve this system

$$\begin{bmatrix} 0 & 1 & 3 \\ 2 & 3 & 4 \\ -2 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad (*)$$

for x, y, and z. The calculation is this.

$$\left[\begin{array}{ccc|c} 0 & 1 & 3 & a \\ 2 & 3 & 4 & b \\ -2 & -1 & 2 & c \end{array}\right] \xrightarrow{\rho_1 \leftrightarrow \rho_2} \xrightarrow{\rho_1+\rho_3} \xrightarrow{-2\rho_2+\rho_3}$$

$$\xrightarrow{(1/2)\rho_1} \xrightarrow{-(3/2)\rho_2+\rho_1} \left[\begin{array}{ccc|c} 1 & 0 & -5/2 & -(3/2)a+(1/2)b \\ 0 & 1 & 3 & a \\ 0 & 0 & 0 & -2a+b+c \end{array}\right]$$

(i) The dimension of the domain space is the number of columns, $n = 3$, and the dimension of the codomain space is the number of rows, $m = 3$.

(ii) There are codomain triples

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$$

for which the system does not have a solution. Specifically, row 3 of the reduced row-echelon form shows that the system has a solution if and only if $-2a + b + c = 0$. Solving this equation, using the free variables b and c gives

$$\mathscr{R}(h) = \{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \mid a = (b+c)/2 \} = \{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} b + \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix} c \mid b, c \in \mathbb{R} \}$$

The second description describes the range as the span of a basis. Alternatively, one can get a (different) basis of the range as the basis of the column space of the matrix. From the calculation of the reduced row echelon form, it follows that a basis of the column space of the matrix is the first two columns. The map's rank is the range's dimension, 2

(iii) Setting $a = b = c = 0$ in the calculation gives infinitely many solutions. Parametrizing using the free variable z leads to this description of the null-space.

$$\mathscr{N}(h) = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid y = -3z \text{ and } x = (5/2)z \} = \{ \begin{bmatrix} 5/2 \\ -3 \\ 1 \end{bmatrix} z \mid z \in \mathbb{R} \}$$

The nullity is the dimension of that null space, 1.

(iv) The map is not onto because the range space is not all of the codomain. The map is not one-to-one because the null space is not trivial.

(c) Here is the needed calculation.

$$\left[\begin{array}{cc|c} 1 & 1 & a \\ 2 & 1 & b \\ 3 & 1 & c \end{array}\right] \xrightarrow[-3\rho_1+\rho_3]{-2\rho_1+\rho_2} \xrightarrow{-2\rho_2+\rho_3} \xrightarrow{-\rho_2} \xrightarrow{-\rho_2+\rho_1} \left[\begin{array}{cc|c} 1 & 0 & -a+b \\ 0 & 1 & 2a-b \\ 0 & 0 & a-2b+c \end{array}\right]$$

(i) The domain has dimension $m = 2$ while the codomain has dimension $n = 3$.

(ii) The range is this subspace of the codomain.

$$\mathscr{R}(h) = \{ \begin{bmatrix} 2b-c \\ b \\ c \end{bmatrix} \mid b,c \in \mathbb{R} \} = \{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} b + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} c \mid b,c \in \mathbb{R} \}$$

As in the previous case, for an alternative description of the range space, one can use a basis of the column space of the 3×2 matrix, which in this case is the two columns of the matrix, since they are linearly independent. The rank is 2.

(iii) The null space is the trivial subspace of the domain.

$$\mathscr{N}(h) = \{ \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \}$$

The nullity is 0.

(iv) The map is not onto, and is one-to-one.

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2. Verify that the map $h\colon \mathbb{R}^n \to \mathbb{R}^m$ represented by this matrix with respect to the standard bases

$$\begin{bmatrix} 2 & 1 & 0 \\ 3 & 1 & 1 \\ 7 & 2 & 1 \end{bmatrix}$$

is an isomorphism.

▶ **Solution.** We have shown that, given a $m \times n$ matrix H and vector spaces V, W of dimension n and m, if we fix bases $B \subseteq V$ and $D \subseteq W$ then the map $h\colon V \to W$ defined by $\mathrm{Rep}_{B,D}(h) = H$ is linear (Theorem Three.III.2.2). So it only remains to show that the map is onto and one-to-one. For that we don't need to augment the matrix with a, b, and c; this calculation

$$\begin{bmatrix} 2 & 1 & 0 \\ 3 & 1 & 1 \\ 7 & 2 & 1 \end{bmatrix} \xrightarrow[-(7/2)\rho_1+\rho_3]{-(3/2)\rho_1+\rho_2} \xrightarrow{-3\rho_2+\rho_3} \xrightarrow[\substack{-2\rho_2 \\ -(1/2)\rho_3}]{(1/2)\rho_1} \xrightarrow{2\rho_3+\rho_2} \xrightarrow{-(1/2)\rho_2+\rho_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

shows that for each codomain vector there is one and only one associated domain vector. ◀

3. Let the homomorphism $h : \mathbb{R}^3 \to \mathcal{P}_2$ be given by

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \mapsto (a+b)x^2 + (2a+2b)x + c.$$

For the bases $\mathcal{B} = \langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rangle$ of $\mathbb{R}^3$ and $\mathcal{C} = \langle 1+x,\, 1-x,\, x^2 \rangle$ of $\mathcal{P}_2$, find the matrix representation $\mathrm{Rep}_{\mathcal{B},\mathcal{C}}(h)$ of h with respect to the bases $\mathcal{B}$ and $\mathcal{C}$.

▶ **Solution.** Note that

$$h\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = 2x^2+4x+1 = \frac{5}{2}(1+x) - \frac{3}{2}(1-x) + 2x^2,$$

$$h\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) = 2x^2+2x+1 = \frac{3}{2}(1+x) - \frac{1}{2}(1-x) + x^2,$$

$$h\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = 0x^2+0x+1 = \frac{1}{2}(1+x) + \frac{1}{2}(1-x) + 0x^2.$$

Thus,

$$\begin{aligned}\mathrm{Rep}_{\mathcal{B},\mathcal{C}}(h) &= \left[\mathrm{Rep}_{\mathcal{C}}(h\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)) \quad \mathrm{Rep}_{\mathcal{C}}(h\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right)) \quad \mathrm{Rep}_{\mathcal{C}}(h\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right))\right] \\ &= \begin{bmatrix}\frac{5}{2} & \frac{3}{2} & \frac{1}{2}\\ -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2}\\ 2 & 1 & 0\end{bmatrix}.\end{aligned}$$

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4. Let $h : \mathbb{R}^3 \to \mathcal{P}_2$ be the homomorphism represented with respect to the bases $\mathcal{E}_3$ of $\mathbb{R}^3$ and $\mathcal{C} = \langle 1,\, 1+x^2,\, x\rangle$ by the matrix

$$H = \begin{bmatrix}1 & 3 & 1\\ 0 & 1 & 1\\ 1 & 0 & -2\end{bmatrix}.$$

(a) Find $h(\vec{v})$ for $\vec{v} = \begin{bmatrix}2\\-3\\1\end{bmatrix}$.

(b) Find $h(\vec{w})$ for the general vector $\vec{w} = \begin{bmatrix}a\\b\\c\end{bmatrix}$.

(c) Determine if $1-3x+x^2$ and $3-x+x^2$ are in the range of h.

▶ **Solution.** (a) $\mathrm{Rep}_{\mathcal{C}}(h(\vec{v})) = H\mathrm{Rep}_{\mathcal{E}_3}(\vec{v}) = \begin{bmatrix}1 & 3 & 1\\ 0 & 1 & 1\\ 1 & 0 & -2\end{bmatrix}\begin{bmatrix}2\\-3\\1\end{bmatrix} = \begin{bmatrix}-6\\-2\\0\end{bmatrix}$, so

$$h(\vec{v}) = -6\cdot 1 - 2(1+x^2) + 0x = -8 - 2x^2.$$

(b) $\mathrm{Rep}_{\mathcal{C}}(h(\vec{w})) = H\mathrm{Rep}_{\mathcal{E}_3}(\vec{w}) = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+3b+c \\ b+c \\ a-2c \end{bmatrix}$, so

$h(\vec{w}) = (a+3b+c)\cdot 1 + (b+c)(1+x^2) + (a-2c)x = (a+4b+2c) + (b+c)x^2 + (a-2c)x.$

(c) $p(x)$ is in the range of h if and only if $\mathrm{Rep}_{\mathcal{C}}(p(x))$ is in the column space of H. Suppose $\mathrm{Rep}_{\mathcal{C}}(p(x)) = \begin{bmatrix} r \\ s \\ t \end{bmatrix}$. To determine if this is in the column space of H do the following row reduction calculation:

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & r \\ 0 & 1 & 1 & s \\ 1 & 0 & -2 & t \end{array}\right] \xrightarrow{-\rho_1+\rho_3} \xrightarrow{3\rho_2+\rho_3} \xrightarrow{-3\rho_2+\rho_1} \left[\begin{array}{ccc|c} 1 & 0 & -2 & r-3s \\ 0 & 1 & 1 & s \\ 0 & 0 & 0 & t-r+3s \end{array}\right]$$

Thus, $\begin{bmatrix} r \\ s \\ t \end{bmatrix}$ is in the column space of H if and only if $t - r + 3s = 0$. Since $\mathrm{Rep}_{\mathcal{C}}(1 - 3x + x^2) = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$ and $t - r + 3s = -3 - 0 + 3 \cdot 1 = 0$ it follows that $1 - 3x + x^2$ is in the range of h. In particular, $h(\vec{w}) = 1 - 3x + x^2$ for $\vec{w} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$ from the formula in part (b). Since $\mathrm{Rep}_{\mathcal{C}}(3 - x + x^2) = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ and $t - r + 3s = -1 - 2 + 3 = 0$ it follows that $3 - x + x^2$ is in the range of h. In particular, $h(\vec{w}) = 3 - x + x^2$ for $\vec{w} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ from the formula in part (b).

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