In all homework problems, it is not sufficient to show only the answers. *You must show your work.* These exercises are based on Chapter Three.I and Three.II from the text.

1. Use these matrices for this exercise.

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & -2 & 3 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix}$$

(a) Find $-3A$, $2A + 3C$ and $2B - 5D$, or state "not defined."

▶ **Solution.**

$$-3A = \begin{bmatrix} 3 * 9 & -3 & \\ 0 & 3 & 6 \end{bmatrix}, \qquad 2A + 3C \text{ not defined}, \qquad 2B - 5D = \begin{bmatrix} 4 & 4 \\ -22 & 0 \end{bmatrix}$$

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(b) There are 16 possible ways to multiply two of these matrices together, including multiplying a matrix by itself. Which of the possible matrix products are not defined?

▶ **Solution.** A^2, AB, AD, BC, CA, CB, CD, DC are not defined. ◀

(c) Compute AB and AC, or state "not defined."

▶ **Solution.** AB is not defined. $AC = \begin{bmatrix} 1 & 6 & 1 \\ 5 & -3 & 7 \end{bmatrix}$. ◀

2. Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 9 \\ -3 & k \end{bmatrix}$. What value of k, if any, will make $AB = BA$?

▶ **Solution.** $AB = \begin{bmatrix} -7 & 18 + 3k \\ -4 & -9 + k \end{bmatrix}$ and $BA = \begin{bmatrix} -7 & 12 \\ -6 - k & -9 + k \end{bmatrix}$. Thus $AB = BA$ if and only if $-7 = -7$, $18 + 3k = 12$, $-4 = -6 - k$, and $-9 + k = -9 + k$. The only k that satisfies all these is $k = -2$. ◀

3. Let $A = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix}$, and $C = \begin{bmatrix} -3 & -5 \\ 2 & 1 \end{bmatrix}$. Verify that $AB = AC$, even though $B \neq C$.

▶ **Solution.** $AB = \begin{bmatrix} -21 & -21 \\ 7 & 7 \end{bmatrix}$ while $AC = \begin{bmatrix} -21 & -21 \\ 7 & 7 \end{bmatrix}$. Thus $AB = AC$ but $B \neq C$. ◀

4. Show how to use matrix multiplication to bring this matrix to reduced row echelon form.

$$\begin{bmatrix} 2 & 1 & 4 & -1 \\ 1 & 3 & 10 & 8 \\ -1 & 2 & 3 & 7 \end{bmatrix}$$

► **Solution.** The reduction of the matrix to reduced row-echelon form is accomplished by the following sequence of row operations applied in order to the matrix, which we call A:

$$A \xrightarrow{\rho_1 \leftrightarrow \rho_2} \xrightarrow[-2\rho_1+\rho_2]{\rho_1+\rho_3} \xrightarrow{\rho_2+\rho_3} \xrightarrow{-\frac{1}{3}\rho_3} \xrightarrow[16\rho_3+\rho_2]{-10\rho_3+\rho_1} \xrightarrow{-\frac{1}{5}\rho_2} \xrightarrow{-3\rho_2+\rho_1} \begin{bmatrix} 1 & 0 & 0 & -\frac{37}{15} \\ 0 & 1 & 0 & \frac{19}{15} \\ 0 & 0 & 1 & \frac{2}{3} \end{bmatrix} = B$$

Since each row operation is obtained by multiplication of an elementary matrix, the reduced row-echelon form B of A is

$$B = C_{2,1}(-3)M_2(-\frac{1}{5})C_{3,2}(16)C_{3,1}(-10)M_3(-\frac{1}{3})C_{2,3}(1)C_{1,2}(-2)C_{1,3}(1)P_{1,2}A$$

Since there is not a unique sequence of elementary row operations that reduce A to reduced row-echelon form, there are many other multiplications of A on the left by elementary matrices to produce B. ◄

5. Find the inverse or show the inverse does not exist for each of the following matrices.

(a) $A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$ (b) $B = \begin{bmatrix} 1 & 2 & 1 \\ -4 & -7 & 3 \\ -2 & -6 & 4 \end{bmatrix}$

► **Solution.** (a) Apply elementary row operations to the augmented matrix $\begin{bmatrix} A \mid I_3 \end{bmatrix}$ to try to reduce to $\begin{bmatrix} I_3 & A^{-1} \end{bmatrix}$, if possible:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[3\rho_1+\rho_2]{2\rho_1+\rho_3} \xrightarrow{3\rho_2+\rho_3} \xrightarrow[\rho_3+\rho_2]{\rho_3+\rho_1} \xrightarrow{\frac{1}{2}\rho_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{array}\right]$$

Then $A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$

(b) Apply elementary row operations to the augmented matrix $\begin{bmatrix} B \mid I_3 \end{bmatrix}$ to try to reduce to $\begin{bmatrix} I_3 & B^{-1} \end{bmatrix}$, if possible:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -4 & -7 & 3 & 0 & 1 & 0 \\ -2 & -6 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[4\rho_1+\rho_2]{2\rho_1+\rho_3} \xrightarrow{2\rho_2+\rho_3} \xrightarrow{\frac{1}{20}\rho_3}$$

$$\xrightarrow[-7\rho_3+\rho_2]{-\rho_3+\rho_1} \xrightarrow{-2\rho_2+\rho_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -\frac{7}{10} & \frac{13}{20} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{3}{10} & -\frac{7}{20} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{10} & \frac{1}{20} \end{array}\right]$$

Then $B^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{7}{10} & \frac{13}{20} \\ \frac{1}{2} & \frac{3}{10} & -\frac{7}{20} \\ \frac{1}{2} & \frac{1}{10} & \frac{1}{20} \end{bmatrix}$

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6. For each space find the matrix changing a vector representation with respect to B to one with respect to D.

   (a) $V = \mathbb{R}^3$, $B = \mathcal{E}_3$, $D = \langle \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \rangle$

   ▶ **Solution.** Start by computing the effect of the identity function on each element of the starting basis B. Obviously this is the effect.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \xmapsto{\text{id}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \xmapsto{\text{id}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \xmapsto{\text{id}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

   Now represent the three outputs with respect to the ending basis.

$$\text{Rep}_D(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}) = \begin{bmatrix} -2/3 \\ 5/3 \\ -1/3 \end{bmatrix} \quad \text{Rep}_D(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}) = \begin{bmatrix} 1/3 \\ -1/3 \\ 2/3 \end{bmatrix} \quad \text{Rep}_D(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}) = \begin{bmatrix} 1/3 \\ -1/3 \\ -1/3 \end{bmatrix}$$

   Concatenate them into a basis.

$$\text{Rep}_{B,D}(\text{id}) = \begin{bmatrix} -2/3 & 1/3 & 1/3 \\ 5/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \end{bmatrix}$$

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   (b) $V = \mathbb{R}^3$, $B = \langle \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \rangle$, $D = \mathcal{E}_3$

   ▶ **Solution.** One way to find this is to take the inverse of the prior matrix, since it converts bases inthe other direction. Alternatively, we can compute these three

$$\text{Rep}_{\mathcal{E}_3}(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{Rep}_{\mathcal{E}_3}(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{Rep}_{\mathcal{E}_3}(\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

   and put them in a matrix.

$$\text{Rep}_{B,D}(\text{id}) = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 3 & 1 & -1 \end{bmatrix}$$

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(c) $V = \mathcal{P}_2$, $B = \langle x^2, x^2 + x, x^2 + x + 1\rangle$, $D = \langle 2, -x, x^2\rangle$

▶ **Solution.** Representing $\mathrm{id}(x^2)$, $\mathrm{id}(x^2 + x)$, and $\mathrm{id}(x^2 + x + 1)$ with respect to the ending basis gives this.

$$\mathrm{Rep}_D(x^2) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathrm{Rep}_D(x^2 + x) = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \mathrm{Rep}_D(x^2 + x + 1) = \begin{bmatrix} 1/2 \\ -1 \\ 1 \end{bmatrix}$$

Put them together.

$$\mathrm{Rep}_{B,D}(\mathrm{id}) = \begin{bmatrix} 0 & 0 & 1/2 \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

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