In all homework problems, it is not sufficient to show only the answers. *You must show your work.* These exercises are based on Chapter Four.I and Four.III from the text.

1. Compute the determinant of each of the following matrices. Show your work.

$$\text{(a)}\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \qquad \text{(b)}\ \begin{bmatrix} 2 & 0 & 3 & 2 \\ 5 & 1 & 2 & 4 \\ 3 & 0 & 1 & 2 \\ 5 & 3 & 2 & 1 \end{bmatrix} \qquad \text{(c)}\ \begin{bmatrix} 2 & 1 & 1 & 0 & 1 \\ 2 & 1 & 2 & -1 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 1 & 0 & 3 & 1 & 1 \\ 2 & 1 & 1 & 2 & 1 \end{bmatrix}$$

▶ **Solution.** (a) The matrix is reduced to an upper triangular matrix by the following sequence of elementary row operations:

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \xrightarrow[-\rho_2+\rho_1]{} \xrightarrow[\substack{\rho_1+\rho_3 \\ \rho_1+\rho_4}]{\rho_1+\rho_2} \xrightarrow[-2\rho_2+\rho_4]{-2\rho_2+\rho_3} \xrightarrow{-\frac{1}{2}\rho_3+\rho_4} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & -\frac{3}{2} \end{bmatrix} = B$$

Since each of these elementary row operations does not change the determinant, it follows that the determinant of the original matrix A is the determinant of the upper triangular matrix B. The latter determinant is the product of the diagonal entries. Thus

$$\det A = \det B = (-1) \cdot 1 \cdot (-2) \cdot (-\frac{3}{2}) = -3.$$

(b) The matrix is reduced to an upper triangular matrix by the following sequence of elementary row operations:

$$A = \begin{bmatrix} 2 & 0 & 3 & 2 \\ 5 & 1 & 2 & 4 \\ 3 & 0 & 1 & 2 \\ 5 & 3 & 2 & 1 \end{bmatrix} \xrightarrow[\substack{-\frac{3}{2}\rho_1+\rho_3 \\ -\frac{5}{2}\rho_1+\rho_4}]{-\frac{5}{2}\rho_1+\rho_2} \xrightarrow[-2\rho_2+\rho_4]{} \xrightarrow{-\frac{22}{7}\rho_3+\rho_4} \begin{bmatrix} 2 & 0 & 3 & 2 \\ 0 & 1 & -\frac{11}{2} & -1 \\ 0 & 0 & -\frac{7}{2} & -1 \\ 0 & 0 & 0 & -\frac{29}{7} \end{bmatrix} = B$$

Since each of these elementary row operations does not change the determinant, it follows that the determinant of the original matrix A is the determinant of the upper triangular matrix B. The latter determinant is the product of the diagonal entries. Thus

$$\det A = \det B = 2 \cdot 1 \cdot (-\frac{7}{2}) \cdot (-\frac{29}{7}) = 29.$$

(c) The matrix is reduced to an upper triangular matrix by the following sequence of elementary row operations:

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 & 1 \\ 2 & 1 & 2 & -1 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 1 & 0 & 3 & 1 & 1 \\ 2 & 1 & 1 & 2 & 1 \end{bmatrix} \xrightarrow[\substack{-\frac{1}{2}\rho_1+\rho_4 \\ -\rho_1+\rho_4}]{-\rho_1+\rho_2} \xrightarrow{\rho_4\leftrightarrow\rho_2} \xrightarrow[-\rho_3+\rho_4]{} \xrightarrow{-\frac{3}{2}\rho_4+\rho_5} \begin{bmatrix} 2 & 1 & 1 & 0 & 1 \\ 0 & -\frac{1}{2} & -\frac{5}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B$$

Since each of these elementary row operations does not change the determinant, except for the single row swap, it follows that the determinant of the original matrix A is minus the determinant of the upper triangular matrix B. The latter determinant is the product of the diagonal entries. Since one of the diagonal elements is 0, $\det B = 0$, and hence $\det A = -\det B = 0$.

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2. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. If $\det A = 7$, compute the determinant of each of the following matrices.

(a) $\begin{bmatrix} a & b & c \\ 2d & 2e & 2f \\ 5g & 5h & 5i \end{bmatrix}$ (b) $\begin{bmatrix} 3d & 3e & 3f \\ a & b & c \\ g & h & i \end{bmatrix}$ (c) $\begin{bmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{bmatrix}$

▶ **Solution.** (a)

$$\begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ 5g & 5h & 5i \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{vmatrix} = 2\cdot 5\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2\cdot 5\cdot 7 = 70.$$

(b)

$$\begin{vmatrix} 3d & 3e & 3f \\ a & b & c \\ g & h & i \end{vmatrix} = 3\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} = 3\cdot(-1)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -21.$$

(c)

$$\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 14.$$

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3. Compute $\det B^5$ where $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

▶ **Solution.** First compute $\det B$: Use Laplace expansion along row 1 to get

$$\det B = 1\cdot\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} - 0\cdot\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + 1\cdot\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -3 + 1 = -2.$$

Then the product rule for determinants gives $\det B^5 = (\det B)^5 = (-2)^5 = -32$. ◀

4. Verify that $\det(AB) = \det A \det B$ for each of the following pairs of matrices.

(a) $A = \begin{bmatrix} 3 & 0 \\ 6 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 5 & 4 \end{bmatrix}$

(b) $A = \begin{bmatrix} 3 & 6 \\ -1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 2 \\ -1 & -1 \end{bmatrix}$

▶ **Solution.** (a) $\det A = 3$, $\det B = 8$ and $AB = \begin{bmatrix} 6 & 0 \\ 17 & 4 \end{bmatrix}$. Thus, $\det(AB) = 6 \cdot 4 = 24 = 3 \cdot 8 = \det A \det B$.

(b) $\det A = 3 \cdot (-2) - (-1) \cdot 6 = 0$, $\det B = 4 \cdot (-1) - (-1) \cdot 2 = -2$ and $AB = \begin{bmatrix} 6 & 0 \\ -2 & 0 \end{bmatrix}$. Thus, $\det(AB) = 0 = \det A \det B$.

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5. Let A and B be 3×3 matrices, with $\det A = 4$ and $\det B = -3$. Use properties of determinants to compute:

(a) $\det AB$ (b) $\det 5A$ (c) $\det B^{-1}$ (d) $\det A^3$ (e) $\det BAB^{-2}$

▶ **Solution.** (a) $\det AB = \det A \det B = 4 \cdot (-3) = -12$.

(b) $\det 5A = 5^3 \det A = 5^3 \cdot 4 = 125 \cdot 4 = 500$. The first equality follows from the property $\det kA = k^n \det A$ if A is an $n \times n$ matrix.

(c) $\det B^{-1} = 1/\det B = -1/3$.

(d) $\det A^3 = (\det A)^3 = 4^3 = 64$.

(e) $\det BAB^{-2} = (\det B)(\det A)(\det B)^{-2} = (-3) \cdot 4 \cdot (-3)^{-2} = -4/3$.

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6. For each of the following matrices, determine all of the values of x for which the matrix is not invertible.

(a) $A = \begin{bmatrix} 2-x & 1 \\ 4 & 2-x \end{bmatrix}$ (b) $B = \begin{bmatrix} 4-x & -4 & -4 \\ 2 & -2-x & -4 \\ 3 & -3 & -4-x \end{bmatrix}$

▶ **Solution.** (a) A is not invertible if and only if $\det A = 0$. In this case, $\det A = (2-x)^2 - 4 = x^2 - 4x = 0$ if and only if $x = 0$ or $x = 4$.

(b) B is not invertible if and only if $\det B = 0$. Use Laplace expansion along the first row to compute $\det B$:

$$\begin{aligned}\det B &= (4-x)((-2-x)(-4-x)-(-3)(-4)) - (-4)(2(-4-x)-3(-4)) \\ &\quad + (-4)(2(-3)-3(-2-x)) \\ &= (4-x)(x^2+6x-4)+4(4-5x) \\ &= 4x^2+24x-16-x^3-6x^2+4x+16-20x \\ &= -x^3-2x^2+8x \\ &= (-x)(x^2+2x-8) = (-x)(x-2)(x+4).\end{aligned}$$

Thus, B is not invertible precisely for $x = 0$, $x = 2$ and $x = -4$.

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