Math 2085 Section 1
Final Exam
December 11, 1992

Instructions. Work on your own paper and show all relevant work. Be sure to read each problem carefully and do exactly what is requested, no more and no less. Please turn in the exam paper with your name and student number written legibly below. In addition, circle yes or no according as you would or would not like to have your grade posted. If you want your grade posted, then pick up your identifying code before leaving.

Name:

Student Number:

Post Grade? Yes No

1. Let $A = \begin{bmatrix} 1 & 2 & 1 & 7 \\ 2 & 4 & -1 & 2 \\ 1 & 2 & -2 & -5 \end{bmatrix}$ and let $B = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Then B is the reduced row-echelon matrix associated to A.
   (a) Solve the linear system for which A is the augmented matrix.
   (b) Solve the homogeneous linear system for which A is the coefficient matrix.
   (c) Compute a basis for the nullspace $\mathcal{N}(A)$.
   (d) Compute a basis for the row space $\mathcal{R}(A)$.
   (e) Compute a basis for the row space $\mathcal{C}(A)$.

2. Find conditions that b_1, b_2, and b_3 must satisfy for the system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= b_1 \\ 2x_1 - x_2 + 3x_3 &= b_2 \\ 5x_2 - 5x_3 &= b_3 \end{aligned}$$

   to be consistent.

3. Find all values of x for which the matrix

$$A = \begin{bmatrix} x-1 & x-1 & 0 \\ 4 & 2 & 1 \\ x+2 & 1 & x \end{bmatrix}$$

   is *not* invertible.

4. If $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, compute A^{-1}.

5. Let A be a 3×3 matrix with $\det A = 5$. Compute each of the following.

(a) $\det(2A^{-1})$

(b) $\det B$ where B is obtained from A by the following sequence of elementary row operations: $R_1 \leftrightarrow R_2$, $4R_2$, $5R_1 + R_3$.

(c) $\det(AC)$ where $C = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & 5 \\ 0 & 0 & 3 \end{bmatrix}$.

(d) $A(\mathrm{Adj}(A))$.

6. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation. Let $r = \mathrm{Rank}(T)$ and $\nu = \mathrm{nullity}(T)$. Determine whether statements (a) to (d) are:

(I) true for every linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$

(II) true for *some but not all* linear transformations $T : \mathbb{R}^4 \to \mathbb{R}^3$

(III) false for every linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$.

(a) $r \leq 3$, (b) $r = 4$ and $\nu = 0$, (c) $\nu = 2$ and $r = 2$, (d) $\nu \geq 1$.

7. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ be the basis of $\mathbb{R}^2$ with $\mathbf{v}_1 = (1, 1)$ and $\mathbf{v}_2 = (1, -1)$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation with matrix $[T]_{\mathcal{B}} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ with respect to the basis $\mathcal{B}$.

(a) Find the coordinate matrices $[T(\mathbf{v}_1)]_{\mathcal{B}}$ and $[T(\mathbf{v}_2)]_{\mathcal{B}}$.

(b) Find $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$.

(c) Find a formula for $T(x_1, x_2)$ and use this formula to compute $T(3, -1)$.

8. (a) Verify that $\mathcal{B} = \{x + 1, x - 1, x^2 + 1\}$ is a basis of the vector space P_2 of polynomials of degree ≤ 2.

(b) Let $p(x) = 2x^2 - x + 1$. Compute the coordinate matrix $[p(x)]_{\mathcal{B}}$.

9. Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$. Determine if $\mathbf{b}$ is in the column space of A. If it is, express $\mathbf{b}$ as a linear combination of the column vectors of A.

10. Let W be the subspace of $\mathbb{R}^3$ spanned by $\{(1, -1, 1), (1, 2, 1)\}$.

(a) Verify that $\left\{\frac{1}{\sqrt{3}}(1, -1, 1), \frac{1}{\sqrt{6}}(1, 2, 1)\right\}$ is an orthonormal basis of W.

(b) Let $\mathbf{u} = (1, 2, 3)$. Compute the projection of $\mathbf{u}$ on W ($\mathrm{proj}_W \mathbf{u}$).

11. Let $A = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 5 & -2 \\ 0 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 0 & 5 \end{bmatrix}$. Then A and B both have the same characteristic polynomial:

$$\det(\lambda I - A) = \det(\lambda I - B) = (\lambda - 3)^2(\lambda - 5).$$

(a) What are the eigenvalues of A and B?

(b) Find an invertible matrix P such that $P^{-1}AP = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

(c) Verify that B is *not* diagonalizable.