Math 2085 Section 1
Final Exam
December 11, 1992

Instructions. Work on your own paper and show all relevant work. Be sure to read each problem carefully and do exactly what is requested, no more and no less. Please turn in the exam paper with your name and student number written legibly below. In addition, circle yes or no according as you would or would not like to have your grade posted. If you want your grade posted, then pick up your identifying code before leaving.

Name:

Student Number:

Post Grade? Yes No

1. Let $A = \begin{bmatrix} 1 & 2 & 1 & 7 \\ 2 & 4 & -1 & 2 \\ 1 & 2 & -2 & -5 \end{bmatrix}$ and let $B = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Then B is the reduced row-echelon matrix associated to A.

   (a) Solve the linear system for which A is the augmented matrix.

   ▶ **Solution.** The only free variable is x_2 so the solution set of $A'x = b$ where $[A'\ b] = A$ is

   $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3-2t \\ t \\ 4 \end{bmatrix} \quad \text{where } t \text{ is arbitrary.}$$

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   (b) Solve the homogeneous linear system for which A is the coefficient matrix.

   ▶ **Solution.** In this case there are two free variables, namely x_2 and x_4 so the solution set of the equation $Ax = 0$ is

   $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x4 \end{bmatrix} = \begin{bmatrix} -2s-3t \\ s \\ -4t \\ t \end{bmatrix} \quad \text{where } s \text{ and } t \text{ are arbitrary.}$$

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   (c) Compute a basis for the nullspace $\mathcal{N}(A)$.

   ▶ **Solution.** From part (b) a basis is $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ where

   $$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \end{bmatrix}.$$

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(d) Compute a basis for the row space $\mathcal{R}(A)$.

▶ **Solution.** A basis for the row space of A consists of the nonzero rows of B. ◀

(e) Compute a basis for the row space $\mathcal{C}(A)$.

▶ **Solution.** A basis for the column space of A consists of the columns of A corresponding to the columns of B that contain the leading ones, that is, columns 1 and 3. Hence a basis of $\mathcal{C}(A)$ is $\{\mathbf{w}_2, \mathbf{w}_2\}$ where

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \text{ and } \mathbf{w}_2 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}.$$

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2. Find conditions that b_1, b_2, and b_3 must satisfy for the system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= b_1 \\ 2x_1 - x_2 + 3x_3 &= b_2 \\ 5x_2 - 5x_3 &= b_3 \end{aligned}$$

to be consistent.

3. Find all values of x for which the matrix

$$A = \begin{bmatrix} x-1 & x-1 & 0 \\ 4 & 2 & 1 \\ x+2 & 1 & x \end{bmatrix}$$

is *not* invertible.

▶ **Solution.** The matrix A is not invertible if and only if $\det A = 0$. But

$$\begin{aligned} \det A &= (x-1)\det \begin{bmatrix} 1 & 1 & 0 \\ 4 & 2 & 1 \\ x+2 & 1 & x \end{bmatrix} \\ &= (x-1)\left(\begin{bmatrix} 2 & 1 \\ 1 & x \end{bmatrix} - \begin{bmatrix} 4 & 1 \\ x+2 & x \end{bmatrix}\right) \\ &= (x-1)((2x-1) - (4x - (x+2)) \\ &= -(x-1)^2. \end{aligned}$$

Hence, A is not invertible if and only if $x = 1$. ◀

4. If $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, compute A^{-1}. **Answer:** $A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix}$.

5. Let A be a 3×3 matrix with $\det A = 5$. Compute each of the following.

   (a) $\det(2A^{-1})$

   ▶ **Solution.** $\det(2A^{-1}) = 2^3 \det(A^{-1}) = 2^3/5 = 8/5$. ◀

   (b) $\det B$ where B is obtained from A by the following sequence of elementary row operations: $R_1 \leftrightarrow R_2$, $4R_2$, $5R_1 + R_3$.

   ▶ **Solution.** The first row operation multiplies $\det A$ by -1, while the second multiplies it by 4, and the third leaves the determinant unchanged. Hence $\det B = -4 \times 5 = -20$. ◀

   (c) $\det(AC)$ where $C = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & 5 \\ 0 & 0 & 3 \end{bmatrix}$.

   ▶ **Solution.** $\det(AC) = (\det A)(\det C) = 5 \times (-6) = -30$. ◀

   (d) $A(\mathrm{Adj}(A))$.

   ▶ **Solution.** By the adjoint theorem, $A(\mathrm{Adj}(A)) = (\det A)I$ and since A is 3×3, it follows that

   $$\det(A(\mathrm{Adj}(A))) = \det(5I) = 5^3 = 125.$$

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6. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation. Let $r = \mathrm{Rank}(T)$ and $\nu = \mathrm{nullity}(T)$. Determine whether statements (a) to (d) are:

   (I) true for every linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$

   (II) true for *some but not all* linear transformations $T : \mathbb{R}^4 \to \mathbb{R}^3$

   (III) false for every linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$.

   (a) $r \leq 3$, (b) $r = 4$ and $\nu = 0$, (c) $\nu = 2$ and $r = 2$, (d) $\nu \geq 1$.

   (a): I; (b): III; (c): II; (d) I.

7. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ be the basis of $\mathbb{R}^2$ with $\mathbf{v}_1 = (1, 1)$ and $\mathbf{v}_2 = (1, -1)$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation with matrix $[T]_{\mathcal{B}} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ with respect to the basis $\mathcal{B}$.

   (a) Find the coordinate matrices $[T(\mathbf{v}_1)]_{\mathcal{B}}$ and $[T(\mathbf{v}_2)]_{\mathcal{B}}$.

   ▶ **Solution.** $[T(\mathbf{v}_1)]_{\mathcal{B}}$ is the first column of $[T]_{\mathcal{B}}$ and $[T(\mathbf{v}_2)]_{\mathcal{B}}$ is the second column of $[T]_{\mathcal{B}}$. Thus,

   $$[T(\mathbf{v}_1)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \text{and} \quad [T(\mathbf{v}_1)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

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(b) Find $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$.

▶ **Solution.** From the definition of $[T(\mathbf{v}_1)]_{\mathcal{B}}$ it follows that

$$T(\mathbf{v}_1) = 2 \cdot \mathbf{v}_1 + 0 \cdot \mathbf{v}_2 = (2,\, 2).$$

Similarly,

$$T(\mathbf{v}_1) = 1 \cdot \mathbf{v}_1 + 1 \cdot \mathbf{v}_2 = (1,\, 1) + (1,\, -1) = (2,\, 0).$$

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(c) Find a formula for $T(x_1,\, x_2)$ and use this formula to compute $T(3,\, -1)$.

▶ **Solution.** Writing

$$(x_1,\, x_2) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = c_1(1,\, 1) + c_2(1,\, -1) = (c_1 + c_2,\, c_1 - c_2)$$

implies that

$$\begin{aligned} x_1 &= c_1 + c_2 \\ x_2 &= c_1 - c_2. \end{aligned}$$

Solving for c_1 and c_1 gives $c_1 = \frac{1}{2}(x_1 + x_2)$ and $c_2 = \frac{1}{2}(x_1 - x_2)$. Then

$$\begin{aligned} T(x_1,\, x_2) &= T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) \\ &= c_1 T(\mathbf{v}_1) + c_2 T(\mathbf{v}_2) \\ &= \frac{1}{2}(x_1 + x_2)(2,\, 2) + \frac{1}{2}(x_1 - x_2)(2,\, 0) \\ &= (2x_1,\, x_1 + x_2). \end{aligned}$$

Hence, $T(3,\, -1) = (6,\, 2)$. ◀

8. (a) Verify that $\mathcal{B} = \{x + 1,\, x - 1,\, x^2 + 1\}$ is a basis of the vector space P_2 of polynomials of degree ≤ 2.

▶ **Solution.** Since P_2 is a vector space of dimension 3 (with standard basis $\{1,\, x,\, x^2\}$) it is only necessary to check that the vectors in $\mathcal{B}$ are linearly independent to guarantee that it is a basis. Thus, suppose there is a linear dependence relation

$$c_1(x + 1) + c_2(x - 1) + c_3(x^2 + 1) = 0.$$

Expanding this out gives

$$(c_1 - c_2 + c_3) + (c_1 + c_2)x + c_3x^2 = 0,$$

which is only possible if the coefficients of 1, x, and x^2 are all zero, which leads to the system of equations

$$\begin{aligned} c_1 - c_2 + c_3 &= 0 \\ c_1 + c_2 &= 0 \\ c_3 &= 0. \end{aligned}$$

The third equation implies that $c_3 = 0$, and then the first two equations give $c_1 = c_2 = 0$. Hence $\mathcal{B}$ is linearly independent, and since it consists of 3 vectors in a 3-dimensional vector space, it follows that $\mathcal{B}$ is a basis of P_2. ◀

(b) Let $p(x) = 2x^2 - x + 1$. Compute the coordinate matrix $[p(x)]_{\mathcal{B}}$.

▶ **Solution.** Write $p(x) = c_1(x+1) + c_2(x-1) + c_3(x^2+1)$ and solve for c_1, c_2 and c_3 to get $c_1 = -1$, $c_2 = 0$, and $c_3 = 2$. Hence,

$$[p(x)]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

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9. Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$. Determine if $\mathbf{b}$ is in the column space of A. If it is, express $\mathbf{b}$ as a linear combination of the column vectors of A.

▶ **Solution.** $\mathbf{b}$ is in the column space of A if and only if the system of equations $Ax = \mathbf{b}$ is consistent. The augment matrix of this system is

$$[A\ \mathbf{b}] = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 1 & 1 & -1 & 0 \\ -1 & -1 & 1 & 0 \end{bmatrix}$$

which can be row reduced to

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since this matrix has no row of the form $\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$ the system of equations is consistent and $\mathbf{b}$ is in the column space of A. Notice that in the row reduced matrix, the last column is the difference between column 1 and column 2. This same relation holds in the first matrix, i.e.,

$$\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}.$$

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10. Let W be the subspace of $\mathbb{R}^3$ spanned by $\{(1, -1, 1), (1, 2, 1)\}$.

(a) Verify that $\left\{\frac{1}{\sqrt{3}}(1, -1, 1), \frac{1}{\sqrt{6}}(1, 2, 1)\right\}$ is an orthonormal basis of W.

▶ **Solution.** The two vectors have dot product 0 and each has length 1, so they form an orthonormal basis ◀

(b) Let $\mathbf{u} = (1,\ 2,\ 3)$. Compute the projection of $\mathbf{u}$ on W ($\text{proj}_W \mathbf{u}$).

▶ **Solution.** $\text{proj}_W \mathbf{u} = \frac{2}{\sqrt{3}}(1,\ -1,\ 1) + \frac{8}{\sqrt{6}}(1,\ 2,\ 1)$. ◀

11. Let $A = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 5 & -2 \\ 0 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 0 & 5 \end{bmatrix}$. Then A and B both have the same characteristic polynomial:

$$\det(\lambda I - A) = \det(\lambda I - B) = (\lambda - 3)^2(\lambda - 5).$$

(a) What are the eigenvalues of A and B?

▶ **Solution.** The eigenvalues of both A and B are $\lambda = 3$ and $\lambda = 5$. ◀

(b) Find an invertible matrix P such that $P^{-1}AP = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

▶ **Solution.** The columns of P need to be a basis of $\mathbb{R}^3$ consisting of eigenvector of A, so we find all eigenvectors:

$\lambda = 5$: For this eigenvalue, the matrix $\lambda I - A = 5I - A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}$, which row reduces to $\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Hence the eigenvectors of A with eigenvalue 5 are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \quad \text{where } t \neq 0.$$

$\lambda = 3$: For this eigenvalue, the matrix $\lambda I - A = 3I - A = \begin{bmatrix} 0 & -1 & 1 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, which row reduces to $\begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. This is a rank 1 matrix so the dimension of the nullspace is 2. Hence the eigenvectors of A with eigenvalue 3 are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ t \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \text{where } s \neq 0 \text{ and } t \neq 0.$$

Thus, the matrix P will be formed by using an eigenvector of A with eigenvalue 5 as column 1 and columns 2 and 3 will be a basis for the eigenspace of A for eigenvalue 3. Hence,

$$P = \begin{bmatrix} \frac{1}{2} & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

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(c) Verify that B is *not* diagonalizable.

▶ **Solution.** B is diagonalizable if and only if there is a basis of $\mathbb{R}^3$ consisting of eigenvector of B, so we find all eigenvectors to see if there are 3 linearly independent eigenvectors.

$\lambda = 5$: For this eigenvalue, the matrix $\lambda I - B = 5I - B = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, which row reduces to $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Hence the eigenvectors of B with eigenvalue 5 are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \quad \text{where } t \neq 0.$$

$\lambda = 3$: For this eigenvalue, the matrix $\lambda I - B = 3I - B = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix}$, which row reduces to $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Hence the eigenvectors of B with eigenvalue 3 are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{where } s \neq 0.$$

The maximum number of linearly independent eigenvectors for B is 2, so there is not a basis of $\mathbb{R}^3$ consisting of eigenvectors of B. Hence B is not diagonalizble. ◀