Instructions. Answer each of the questions on your own paper, and be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper. There are 10 problems, with a total of 125 points possible. The points for each part of each problem are listed in parentheses.

1. Let $A = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 1 & 4 & -7 & 3 & -2 \\ 1 & 5 & -9 & 5 & -9 \\ 0 & 3 & -6 & 2 & -1 \end{bmatrix}$ and let $R = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Then R is the reduced row-echelon matrix associated to A.
   (a) **(5)** Solve the linear system for which A is the augmented matrix.
   (b) **(5)** Solve the homogeneous linear system for which A is the coefficient matrix.
   (c) **(2)** What is the rank of A?
   (d) **(2)** What is the nullity of A?
   (e) **(5)** Compute a basis for $\mathrm{Ker}(A)$, the kernel of A.
   (f) **(5)** Compute a basis for $\mathrm{Im}(A)$, the image of A.

2. (a) **(5)** Find the value(s) of k for which the vector $\vec{v} = \begin{bmatrix} 1 \\ 6 \\ k \end{bmatrix}$ is in the subspace of $\mathbb{R}^3$ spanned by $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$, where

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \text{ and } \vec{v}_3 = \begin{bmatrix} 4 \\ 4 \\ 12 \end{bmatrix}.$$

   (b) **(5)** For each value of k found above, write $\vec{v}$ as a linear combination of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$.

3. (a) **(5)** Complete the following definition: A function $T : V \to W$ from a vector space V to a vector space W is a *linear transformation* if ____________________.
   (b) **(5)** Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that $T(\vec{v}) = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $T(\vec{w}) = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$, where $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Compute $T(\vec{u})$, for $\vec{u} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

4. (a) **(5)** Complete the following definition: A nonempty subset W of a vector space V is a *subspace* of V if ____________________.
   (b) **(5)** Let W be the line in $\mathbb{R}^2$ whose equation is $x_2 = 2x_1 + 1$. Using your definition in part (a), determine (with justification) whether W is a subspace of $\mathbb{R}^2$.
   (c) **(5)** Let U be the subset $U = \{f(x) \in \mathcal{P}_2 \mid f(1) = 0\}$, where $\mathcal{P}_2$ the vector space of all polynomial of degree at most 2. Using your definition in part (a), determine (with justification) whether U is a subspace of $\mathcal{P}_2$.

5. **(10)** Compute the inverse of the matrix $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 1 & 2 \end{bmatrix}$.

6. **(8)** Let A be the 3×3 matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ x & y & z \\ 3 & 5 & 7 \end{bmatrix}$. Assuming that $\det(A) = 8$, compute the determinant of the following matrix

$$B = \begin{bmatrix} x & y & z \\ 0 & 1 & 1 \\ -12 + 3x & -21 + 3y & -29 + 3z \end{bmatrix}.$$

7. Let $A = \begin{bmatrix} -5 & 2 \\ -12 & 5 \end{bmatrix}$.

   (a) **(10)** Find the eigenvalues and compute an eigenvector for each eigenvalue.

   (b) **(5)** Find, if possible, an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. If not possible, explain why.

8. If A is a 3×3 matrix, determine if each of the following statements about A is true (**T**) or false (**F**).

   (a) **(2)** If 2 is an eigenvalue for A then $A - 2I_3$ is invertible.

   (b) **(2)** If $A^2 = A$ then $\det(A) = 0$ or 1.

   (c) **(2)** $\det(kA) = k\det(A)$ for any scalar k.

   (d) **(2)** If the characteristic polynomial of A has roots 1, 2, and 3, then A is diagonalizable.

   (e) **(2)** If A is not invertible, then 0 is an eigenvalues of A.

9. (a) **(5)** If A is a 26×37 matrix with $\dim \operatorname{Ker}(A) = 13$, what is the rank of A?

   (b) **(5)** Consider a linear system $B\vec{x} = \vec{b}$ where B is a 7×4 matrix with $\operatorname{Ker}(B) = \left\{\vec{0}\right\}$. How many solutions can this system have?

10. Let $\mathcal{B} = (p_1(x), p_2(x), p_3(x))$, where

$$p_1(x) = 1, \quad p_2(x) = x, \quad p_3(x) = x^2$$

be the standard basis of P_2, the vector space of polynomials of degree ≤ 2 and let $T : \mathcal{P}_2 \to \mathcal{P}_2$ be the linear transformation defined by $T(p(x)) = 2p'(x) - 3p(x)$.

   (a) **(5)** Find the matrix $[T]_{\mathcal{B}}$ of T with respect to the basis $\mathcal{B}$.

   (b) **(5)** Compute $\det(T)$.

   (c) **(5)** Is T an isomorphism? Why or why not?