Math 2085 Final Exam July 26, 2000

Instructions. Answer each of the questions on your own paper. Be sure to show your work so that partial credit can be adequately assessed.

Please *print* your name and student number in the space provided below, and turn in this sheet with your solution papers. Fold your paper so that your name appears on the outside.

Name:

Student Number:

1. [15 Pts] The matrices A and B shown below are row equivalent.

$$A = \begin{bmatrix} 1 & -2 & 4 & -3 & 6 & 5 \\ 2 & -4 & 9 & -8 & 12 & 6 \\ 3 & -6 & 12 & -9 & 17 & 10 \\ -5 & 10 & -9 & -7 & -15 & 6 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -2 & 4 & -3 & 6 & 5 \\ 0 & 0 & 1 & -2 & 0 & -4 \\ 0 & 0 & 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   (a) Find a basis for Null A.

   (b) What is the dimension of the subspace of $\mathbb{R}^4$ spanned by the columns of A?

   (c) Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all $\mathbf{b} \in \mathbb{R}^4$? Why or why not?

2. [15] Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{bmatrix} 0 & 9 \\ -1 & 6 \end{bmatrix}$. If $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ where $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, find the $\mathcal{B}$-matrix $[T]_{\mathcal{B}}$ of the linear transformation T.

3. [15] For each matrix below, determine whether its columns form a linearly independent set. Give reasons for your answers. (Make as few calculations as possible.)

$$\textbf{a.} \begin{bmatrix} 2 & -2 \\ 8 & -4 \\ -4 & 1 \end{bmatrix} \qquad \textbf{b.} \begin{bmatrix} 3 & -5 & 0 \\ -6 & 2 & 4 \\ 9 & -7 & -4 \end{bmatrix} \qquad \textbf{c.} \begin{bmatrix} 2 & -1 & 0 & 4 \\ -6 & 3 & 0 & -7 \\ 4 & 1 & 0 & -1 \end{bmatrix}$$

4. [12] For each matrix in Exercise 3, determine if the columns of the matrix span $\mathbb{R}^3$. Give reasons for your answers. (Make as few calculations as possible.)

5. [12] Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ -3 & 1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 1 \\ -1 \\ 8 \end{bmatrix}$, and define $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$.

   (a) Determine if $\mathbf{y}$ is in the range of T.

   (b) Does T map $\mathbb{R}^2$ onto $\mathbb{R}^3$? Why or why not?

   (c) Is T one-to-one? Why or why not?

6. [12] Complete the following definitions:

   (a) A set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of vectors in a vector space V is *linearly dependent* if ...

   (b) If A is an $m \times n$ matrix and $\mathbf{x}$ is a vector in $\mathbb{R}^n$, then $A\mathbf{x} = \ldots$ (You may use words to complete the definition. If you use symbols, explain them.)

   (c) A *basis* of a vector space V is a ...

   (d) A scalar λ is an *eigenvalue* of a matrix A if ...

7. [15] Let A be an $n \times n$ matrix. List five statements which are equivalent to the statement "A is an invertible matrix." Use the following terms, one in each statement: (i) the equation $A\mathbf{x} = \mathbf{b}$, (ii) determinant, (iii) row equivalent, (iv) eigenvalue, (v) rank.

8. [10] Solve the equation $A(XB^{-1} + C) = D^{-1}$ for X, assuming that A, B, C, D are invertible matrices.

9. [8] Tell how the value of $\operatorname{Rank} A$ is related to $\operatorname{Col} A$ and $\operatorname{Null} A$.

10. [12] In each case, either explain why H is a subspace of $\mathbb{R}^3$ or explain why H is *not* a subspace.

$$\textbf{a. } H = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{array}{rcl} 2x - 5y & = & 7z \\ 4x + 3z & = & 1 \end{array} \right\} \qquad \textbf{b. } H = \left\{ \begin{bmatrix} -s + 2t \\ 3s - t \\ s - 5t \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

$$\textbf{c. } H = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : 3a - 2b = 5c \right\}$$

11. [8] Explain, by reference to an appropriate theorem, why the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 10 \end{bmatrix}$$

is diagonalizable.

12. [16] The matrix $A = \begin{bmatrix} 3 & -3 & 1 \\ 0 & 2 & 0 \\ -2 & 6 & 0 \end{bmatrix}$ has the eigenvalues $\lambda = 1$ and 2. Diagonalize A. That is, find appropriate matrices and write the matrix equation that relates them to A.