Math 2085 Final Exam July 26, 2000

Instructions. Answer each of the questions on your own paper. Be sure to show your work so that partial credit can be adequately assessed.

Please *print* your name and student number in the space provided below, and turn in this sheet with your solution papers. Fold your paper so that your name appears on the outside.

Name:

Student Number:

1. [15 Pts] The matrices A and B shown below are row equivalent.

$$A = \begin{bmatrix} 1 & -2 & 4 & -3 & 6 & 5 \\ 2 & -4 & 9 & -8 & 12 & 6 \\ 3 & -6 & 12 & -9 & 17 & 10 \\ -5 & 10 & -9 & -7 & -15 & 6 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -2 & 4 & -3 & 6 & 5 \\ 0 & 0 & 1 & -2 & 0 & -4 \\ 0 & 0 & 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for Null A.

▶ **Solution.** Putting B in reduced row-echelon form (it is already in row-echelon form) produces

$$R = \begin{bmatrix} 1 & -2 & 0 & -3 & 0 & -9 \\ 0 & 0 & 1 & -2 & 0 & -4 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The nullspace Null(A) is obtained by solving the homogeneous system $A\mathbf{x} = \mathbf{0}$. From the form of the matrix R we see that the free variables are x_2, x_4 and x_6. Hence,

$$\text{Null}(A) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 2s + 3t + 9u \\ s \\ 2t + 4u \\ t \\ -5u \\ u \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + u \begin{bmatrix} 9 \\ 0 \\ 4 \\ 0 \\ -5 \\ 1 \end{bmatrix} = s\mathbf{v}_1 + t\mathbf{v}_2 + u\mathbf{v}_3 \right\}$$

where : s, t, $u \in \mathbb{R}$ are arbitrary. Thus, a basis of Null(A) is $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. ◀

(b) What is the dimension of the subspace of $\mathbb{R}^4$ spanned by the columns of A?

▶ **Solution.** The dimension of the column space of A is the rank of A, which is 3 since there are 3 nonzero rows in the reduced row echelon form R of A. ◀

(c) Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all $\mathbf{b} \in \mathbb{R}^4$? Why or why not?

► **Solution.** Since the dimension of the column space is 3 which is less than the dimension of $\mathbb{R}^4$, the equation is not consistent for all $Bb \in \mathbb{R}^4$, since the column space, which consists of the $\mathbf{b} \in \mathbb{R}^4$ for which the equation $A\mathbf{x} = \mathbf{b}$ is consistent, cannot be all of $\mathbb{R}^4$. ◄

2. [15] Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{bmatrix} 0 & 9 \\ -1 & 6 \end{bmatrix}$.

If $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ where $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, find the $\mathcal{B}$-matrix $[T]_{\mathcal{B}}$ of the linear transformation T.

► **Solution.** $T(\mathbf{b}_1) = A\mathbf{b}_1 = \begin{bmatrix} 0 & 9 \\ -1 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \end{bmatrix} = 3\mathbf{b}_1$ and $T(\mathbf{b}_2) = A\mathbf{b}_2 = \begin{bmatrix} 0 & 9 \\ -1 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \end{bmatrix} = \mathbf{b}_1 + 3\mathbf{b}_2$. Thus,

$$[T]_{\mathcal{B}} = [[T(\mathbf{b}_1)]_{\mathcal{B}}\ [T(\mathbf{b}_2)]_{\mathcal{B}}] = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}.$$

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3. [15] For each matrix below, determine whether its columns form a linearly independent set. Give reasons for your answers. (Make as few calculations as possible.)

a. $\begin{bmatrix} 2 & -2 \\ 8 & -4 \\ -4 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 3 & -5 & 0 \\ -6 & 2 & 4 \\ 9 & -7 & -4 \end{bmatrix}$ **c.** $\begin{bmatrix} 2 & -1 & 0 & 4 \\ -6 & 3 & 0 & -7 \\ 4 & 1 & 0 & -1 \end{bmatrix}$

► **Solution. a.** The columns are linearly independent since neither column is a scalar multiple of the other.

b. The columns are linearly dependent since the determinant is $3(-8+28)+5(24-36) = 60 - 60 = 0$.

c. The columns are linearly dependent since one column is the 0 column. ◄

4. [12] For each matrix in Exercise 3, determine if the columns of the matrix span $\mathbb{R}^3$. Give reasons for your answers. (Make as few calculations as possible.)

► **Solution. a.** Three vectors are needed to span R^3, so the two columns of this matrix cannot span R^3.

b. The three columns are linearly dependent (from Exercise 3.b) and hence they cannot span R^3.

c. Since $\det \begin{bmatrix} 2 & -1 & 4 \\ -6 & 3 & -7 \\ 4 & 1 & -1 \end{bmatrix} = -30 \neq 0$ it follows that columns 1, 2, and 4 of the given matrix are linearly independent, and hence span $\mathbb{R}^3$. ◄

5. [12] Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ -3 & 1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 1 \\ -1 \\ 8 \end{bmatrix}$, and define $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$.

(a) Determine if $\mathbf{y}$ is in the range of T.

▶ **Solution.** Row reducing the augment matrix $[A\ \mathbf{y}] = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & -1 \\ -3 & 1 & 8 \end{bmatrix}$ produces the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -3 \\ 0 & 0 & 1 \end{bmatrix}$. The last row of the matrix shows that the equation $A\mathbf{x} = \mathbf{y}$ is not consistent, which means that $\mathbf{y}$ is not in the range of T. ◀

(b) Does T map $\mathbb{R}^2$ onto $\mathbb{R}^3$? Why or why not?

▶ **Solution.** No, since $\mathbf{y}$ is not in the range of T. ◀

(c) Is T one-to-one? Why or why not?

▶ **Solution.** Yes because the rank of A is 2 so the dimension of the kernel of T is 0. ◀

6. [12] Complete the following definitions:

(a) A set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of vectors in a vector space V is *linearly dependent* if ...

▶ **Solution.** there is a linear combination $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$ with at least one $c_i \neq 0$. ◀

(b) If A is an $m \times n$ matrix and $\mathbf{x}$ is a vector in $\mathbb{R}^n$, then $A\mathbf{x} = \ldots$ (You may use words to complete the definition. If you use symbols, explain them.)

▶ **Solution.** the linear combination of the columns of A using the scalars of $\mathbf{x}$. That is, $A\mathbf{x} = x_1 \operatorname{Col}_1(A) + x_2 \operatorname{Col}_2(A) + \cdots + x_n \operatorname{Col}_n(A)$. ◀

(c) A *basis* of a vector space V is a ...

▶ **Solution.** a linearly independent spanning set. ◀

(d) A scalar λ is an *eigenvalue* of a matrix A if ...

▶ **Solution.** $A\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x} \neq \mathbf{0}$. ◀

7. [15] Let A be an $n \times n$ matrix. List five statements which are equivalent to the statement "*A is an invertible matrix.*" Use the following terms, one in each statement: (i) the equation $A\mathbf{x} = \mathbf{b}$, (ii) determinant, (iii) row equivalent, (iv) eigenvalue, (v) rank.

▶ **Solution.** (i) The equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^n$.

(ii) The determinant of A is not 0.

(iii) A is row equivalent to the identity matrix I_n.

(iv) 0 is not an eigenvalue of A.

(v) The rank of A is n.

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8. [10] Solve the equation $A(XB^{-1} + C) = D^{-1}$ for X, assuming that A, B, C, D are invertible matrices.

▶ **Solution.** $A(XB^{-1} + C) = D^{-1} \implies XB^{-1} + C = A^{-1}D^{-1} \implies XB^{-1} = A^{-1}D^{-1} - C \implies X = (A^{-1}D^{-1} - C)B$ ◀

9. [8] Tell how the value of Rank A is related to Col A and Null A.

▶ **Solution.** $\operatorname{Rank} A = \dim(\operatorname{Col} A)$ and $\operatorname{Rank} A + \dim(\operatorname{Null} A)$ is the number of columns of A. ◀

10. [12] In each case, either explain why H is a subspace of $\mathbb{R}^3$ or explain why H is *not* a subspace.

a. $H = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{array}{rcl} 2x - 5y & = & 7z \\ 4x + 3z & = & 1 \end{array} \right\}$ **b.** $H = \left\{ \begin{bmatrix} -s + 2t \\ 3s - t \\ s - 5t \end{bmatrix} : s, t \in \mathbb{R} \right\}$

c. $H = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : 3a - 2b = 5c \right\}$

▶ **Solution. a.** This H is not a subspace since H does not contain the 0 vector.
b. This H is the span of the two vectors $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \\ -5 \end{bmatrix}$ and hence is a subspace. (The span of a set of vectors is always a subspace.)
c. This set is the solution set of the homogeneous equation $3a - 2b - 5c = 0$ and hence is a subspace. (Solution sets of homogeneous equations are subspaces.) ◀

11. [8] Explain, by reference to an appropriate theorem, why the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 10 \end{bmatrix}$$

is diagonalizable.

► **Solution.** This matrix is triangular and so the eigenvalues are the diagonal entries, 1, 3, 6, 10. Since these are all distinct, the matrix is diagonalizable. (A matrix with distinct eigenvalues is always diagonalizable.) ◄

12. [16] The matrix $A = \begin{bmatrix} 3 & -3 & 1 \\ 0 & 2 & 0 \\ -2 & 6 & 0 \end{bmatrix}$ has the eigenvalues $\lambda = 1$ and 2. Diagonalize A. That is, find appropriate matrices and write the matrix equation that relates them to A.

► **Solution.** It is necessary to find a basis of $\mathbb{R}^3$ consisting of eigenvectors of A and these vectors are used as the columns of a matrix P such that $P^{-1}AP = D$ where D is a diagonal matrix containing the eigenvalues of A in an order matching the columns of P. The eigenvalues are given to be 1 and 2, so we compute the eigenvectors corresponding to these eigenvalues.

($\lambda = 1$): In this case $\lambda I - A = I - A = \begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 0 \\ 2 & -6 & -1 \end{bmatrix}$. This row reduces to $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ which implies that the eigenvectors corresponding to $\lambda = 1$ are $s \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$ for $s \neq 0$.

($\lambda = 2$): In this case $\lambda I - A = 2I - A = \begin{bmatrix} -1 & 3 & -1 \\ 0 & 0 & 0 \\ 2 & -6 & 2 \end{bmatrix}$, which row reduces to $\begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus, the eigenvectors with eigenvalue 2 are all vectors of the form

$$\begin{bmatrix} 3s - t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

where $s \neq 0$ or $t \neq 0$.

Thus, if we let $P = \begin{bmatrix} -1 & 3 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ then $AP = PD$ where $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Hence, $P^{-1}AP = D$. ◄