Instructions. Answer each of the questions on your own paper, and be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper.

1. [**12 Points**] Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 3 \\ -2 & 1 \end{bmatrix}$. Compute each of the products AB and BA that makes sense. If a product does not make sense, state why it does not make sense.

   ► **Solution.** A is a 2×2 matrix and B is a 3×2 matrix. Thus, AB is not defined since the number of columns of A, which is 2, is not the same as the number of rows of B, which is 3. However, BA is defined since the number of columns of B is the same as the number of rows of A. Hence,

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 9 & 0 \\ 1 & -4 \end{bmatrix}.$$

◄

2. [**12 Points**] Give an example of each of the following:

   (a) A 4×3 matrix that is *not* in row-echelon form.

   ► **Solution.** $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ◄

   (b) A 3×4 matrix that is in row-echelon form, but *not* in reduced row-echelon form.

   ► **Solution.** $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ ◄

   (c) A 4×4 matrix that is in reduced row-echelon form and has exactly 3 nonzero rows.

   ► **Solution.** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ◄

   (d) The 4×4 identity matrix I_4.

► **Solution.** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ ◄

3. [**20 Points**] Find all solutions of the following system of linear equations. Be sure to show all your steps!

$$\begin{array}{ccccccccc} x_1 & + & 2x_2 & + & 2x_3 & + & 6x_4 & = & 1 \\ x_1 & - & x_2 & + & 2x_3 & - & 6x_4 & = & 4 \\ x_1 & + & x_2 & + & 2x_3 & + & 2x_4 & = & 2 \end{array}$$

► **Solution.** Use Gauss-Jordan elimination on the augmented matrix A:

$$A = \begin{bmatrix} 1 & 2 & 2 & 6 & 1 \\ 1 & -1 & 2 & -6 & 4 \\ 1 & 1 & 2 & 2 & 2 \end{bmatrix} \begin{array}{c} R_2 \mapsto R_2 - R_1 \\ \longrightarrow \\ R_3 \mapsto R_3 - R_1 \end{array} \begin{bmatrix} 1 & 2 & 2 & 6 & 1 \\ 0 & -3 & 0 & -12 & 3 \\ 0 & -1 & 0 & -4 & 1 \end{bmatrix}$$

$$\begin{array}{c} -\frac{1}{3}R_2 \\ \longrightarrow \\ R_3 \mapsto R_3 - R_2 \end{array} \begin{bmatrix} 1 & 2 & 2 & 6 & 1 \\ 0 & 1 & 0 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} R_1 \mapsto R_1 - 2R_2 \\ \longrightarrow \\ \end{array} \begin{bmatrix} 1 & 0 & 2 & -2 & 3 \\ 0 & 1 & 0 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The last matrix is in reduced row-echelon form and is the augmented matrix of the linear system

$$\begin{array}{ccccccccc} x_1 & & & + & 2x_3 & - & 2x_4 & = & 3 \\ & & x_2 & & & + & 4x_4 & = & -1 \\ & & & & & & 0 & = & 0 \end{array}$$

The solution set of this last system is the same as that of the original system and is given by:

$$\mathcal{S} = \{(3 - 2x_3 + 2x_4,\ -1 - 4x_4,\ x_3,\ x_4) : x_3 \text{ and } x_4 \text{ are arbitrary}\}.$$

◄

4. [**20 Points**] Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$.

   (a) Compute A^{-1}.

▶ **Solution.** Use Gauss-Jordan elimination of the augmented matrix $\begin{bmatrix} A & \mid & I_3 \end{bmatrix}$:

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{array}{c} R_2 \mapsto R_2 - R_1 \\ \longrightarrow \\ R_3 \mapsto R_3 - R_1 \end{array} \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{c} -R_2 \\ \longrightarrow \\ R_3 \mapsto R_3 + R_2 \end{array} \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{bmatrix} \begin{array}{c} -R_3 \\ \longrightarrow \\ \end{array} \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix}$$

$$\begin{array}{c} R_2 \mapsto R_2 - R_3 \\ \longrightarrow \\ R_1 \mapsto R_1 - 3R_3 \end{array} \begin{bmatrix} 1 & 2 & 0 & 1 & -3 & 3 \\ 0 & 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix} \begin{array}{c} R_1 \mapsto R_1 - 2R_2 \\ \longrightarrow \\ \end{array} \begin{bmatrix} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix}$$

Hence,

$$A^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

◀

(b) Using your answer to part (a), solve the system of equations

$$\begin{array}{ccccccc} x_1 & + & 2x_2 & + & 3x_3 & = & 1 \\ x_1 & + & x_2 & + & 2x_3 & = & 0 \\ x_1 & + & x_2 & + & x_3 & = & 1 \end{array}$$

▶ **Solution.**

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}.$$

◀

5. [**14 Points**] Find conditions that b_1, b_2, and b_3 must satisfy for the following system to be consistent:

$$\begin{array}{ccccccc} x_1 & + & 2x_2 & - & 3x_3 & = & b_1 \\ 2x_1 & + & 3x_2 & + & x_3 & = & b_2 \\ 5x_1 & + & 9x_2 & - & 8x_3 & = & b_3 \end{array}$$

▶ **Solution.** Start by trying to row reduce the augmented matrix:

$$\begin{bmatrix} 1 & 2 & -3 & b_1 \\ 2 & 3 & 1 & b_2 \\ 5 & 9 & -8 & b_3 \end{bmatrix} \begin{array}{c} R_2 \mapsto R_2 - 2R_1 \\ \longrightarrow \\ R_3 \mapsto R_3 - 5R_1 \end{array} \begin{bmatrix} 1 & 2 & -3 & b_1 \\ 0 & -1 & 7 & b_2 - 2b_1 \\ 0 & -1 & 7 & b_3 - 5b_1 \end{bmatrix}$$

$$\begin{array}{c} -R_2 \\ \longrightarrow \\ R_3 \mapsto R_3 + R_2 \end{array} \begin{bmatrix} 1 & 2 & -3 & b_1 \\ 0 & 1 & -7 & -b_2 + 2b_1 \\ 0 & 0 & 0 & b_3 - 3b_1 - b_2 \end{bmatrix}$$

In order for the system to be consistent, the last entry in the last column must be 0, and this entry being 0 is also sufficient to be able to complete the row reduction and solve the system. Hence, the system is consistent if and only if $\boxed{3b_1 + b_2 = b_3}$. ◀

6. [**12 Points**] Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 7 \\ 2 & 2 & 7 \end{bmatrix}$.

   (a) Find elementary matrices E_1, E_2, and E_3 so that $E_1E_2E_3A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$.

   ▶ **Solution.** Just row reduce the matrix until it is in the form where only 0 is found below the diagonal, and keep track of the row operations needed. Each such row operation corresponds to a matrix multiplication (on the left) by an elementary matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 7 \\ 2 & 2 & 7 \end{bmatrix} \underset{R_3 \mapsto R_3 - 2R_1}{\overset{R_2 \mapsto R_2 - 2R_1}{\longrightarrow}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & -2 & 1 \end{bmatrix} \underset{R_3 \mapsto R_3 + R_2}{\longrightarrow} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} = B$$

   The elementary matrices corresponding to the 3 row operations performed are:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \mapsto R_2 - 2R_1} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \mapsto R_3 - 2R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = E_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \mapsto R_3 + R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = E_1$$

   Then $B = E_1E_2E_3A$. ◀

   (b) Using your answer to part (a), compute $\det A$.

   ▶ **Solution.** Each of the row operations in Part(a) does not change the determinant. Thus, $\det A = \det B = 4$ since the determinant of a triangular matrix is the product of the diagonal entries. ◀

7. [**10 Points**] Let A be an $n \times n$ invertible matrix. Determine whether each of the following statements is True (**T**) or False (**F**).

(a) $Ax = 0$ has only the solution $x = 0$. **True**

(b) $A^{-1}x = 0$ has infinitely many solutions. **False**

(c) The reduced row-echelon form of A has a row of zeros. **False**

(d) $Ax = b$ has a unique solution for every $n \times 1$ matrix b. **True**

(e) $\det A \neq 0$. **True**