Instructions. Answer each of the questions on your own paper, and be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper.

1. [**12 Points**] Which of the following subsets of the set of 2×2 matrices $M_{2\,2}$ are subspaces of $M_{2\,2}$? Give a (brief) reason in each case.

   (a) S_1 is the set of all matrices of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$.

   ▶ **Solution.** Adding two matrices in S_1 gives another matrix in S_1 and so does scalar multiplication: $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} + \begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix} = \begin{bmatrix} a+a' & b+b' \\ 0 & c+c' \end{bmatrix}$ and $k\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} ka & kb \\ 0 & kc \end{bmatrix}$. Thus S_1 is a subspace. ◀

   (b) S_2 is the set of all matrices of the form $\begin{bmatrix} a & b \\ 2 & c \end{bmatrix}$.

   ▶ **Solution.** $\begin{bmatrix} a & b \\ 2 & c \end{bmatrix} + \begin{bmatrix} a & b \\ 2 & c \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 4 & 2c \end{bmatrix}$. Thus, adding two matrices in S_2 does not produce a matrix in S_2. Thus, S_2 is not a subspace. ◀

2. [**24 Points**] Let $A = \begin{bmatrix} 2 & -4 & 1 & 1 & 0 \\ -1 & 2 & -1 & 1 & 6 \\ 2 & -4 & 0 & 4 & 12 \\ 1 & -2 & 3 & 5 & 30 \end{bmatrix}$ and let $R = \begin{bmatrix} 1 & -2 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Then R is the reduced row-echelon matrix associated to A.

   (a) Compute a basis for the row space $\mathrm{Row}(A)$ of A.

   ▶ **Solution.** A basis for $\mathrm{Row}(A)$ is the 3 nonzero rows of R. ◀

   (b) Compute a basis for the column space $\mathrm{Col}(A)$ of A.

   ▶ **Solution.** A basis for $\mathrm{Col}(A)$ consists of the columns of A corresponding to the columns of R that represent fixed variables. That is, a basis for $\mathrm{Col}(A)$ consists of columns 1, 3, and 4 of A. ◀

   (c) Compute a basis for the nullspace $\mathrm{Null}(A)$ of A.

   ▶ **Solution.** This is obtained by solving $Rx = 0$ and factoring out the free variables $x_2 = s$ and $x_5 = t$. Thus, a basis is

   $$\{(2,\, 1,\, 0,\, 0,\, 0),\, (4,\, 0,\, -3,\, -5,\, 1)\}.$$

   ◀

   (d) What is the rank of A?

▶ **Solution.** The rank of A is the dimension of the row space. Thus the rank is 3. ◀

3. [**20 Points**]

(a) Show that the vectors $\mathbf{v}_1 = (1, 2, 3, 4)$, $\mathbf{v}_2 = (0, 1, 0, -1)$, and $\mathbf{v}_3 = (1, 3, 3, 3)$ form a linearly dependent set in $\mathbb{R}^4$.

▶ **Solution.** The vector equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$ is

$$k_1(1, 2, 3, 4) + k_2(0, 1, 0, -1) + k_3(1, 3, 3, 3) = (0, 0, 0, 0),$$

which gives a system of 4 equations in 3 unknowns:

$$\begin{aligned} k_1 + k_3 &= 0 \\ 2k_1 + k_2 + 3k_3 &= 0 \\ 3k_1 + 3k_3 &= 0 \\ 4k_1 - k_2 + 3k_3 &= 0. \end{aligned}$$

This system has the nontrivial solutions $(k_1, k_2, k_3) = k_3(-1, -1, 1)$ for k_3 arbitrary. In particular there is the nontrivial dependence relation

$$\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0},$$

so the vectors are linearly dependent. ◀

(b) Express each of the vectors as a linear combination of the others.

▶ **Solution.** From the solution to part (a): $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{v}_1 = -\mathbf{v}_2 + \mathbf{v}_3$ and $\mathbf{v}_2 = -\mathbf{v}_1 + \mathbf{v}_3$. ◀

4. [**20 Points**] Let $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ be the basis of $\mathbb{R}^2$ with $\mathbf{v}_1 = (1, 2)$ and $\mathbf{v}_2 = (1, 3)$.

(a) If $\mathbf{v} = (2, 4)$ compute the coordinate vector $(\mathbf{v})_S$.

▶ **Solution.** $(\mathbf{v})_S = (c_1, c_2)$ where $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$. Thus, writing

$$(2, 4) = c_1(1, 2) + c_2(1, 3)$$

gives the equations

$$\begin{aligned} c_1 + c_2 &= 2 \\ 2c_1 + 3c_2 &= 4. \end{aligned}$$

Solving these for c_1, c_2 gives $c_1 = 2$, $c_2 = 0$. Thus $(\mathbf{v})_S = (2, 0)$. ◀

(b) If $(\mathbf{w})_S = (3, -2)$, then find $\mathbf{w}$.

▶ **Solution.** $\mathbf{w} = 3\mathbf{v}_1 - 2\mathbf{v}_2 = 3(1, 2) - 2(1, 3) = (1, 0)$ ◀

5. **[12 Points]** Let A be a 5×7 matrix. and assume that the dimension of the nullspace of A is 4.

   (a) What is the dimension of the column space of A?

   ▶ **Solution.** $\dim \mathrm{Col}(A) = (\#\text{columns}) - \dim(\text{Nullspace } (A)) = 7 - 4 = 3.$ ◀

   (b) What is the rank of A?

   ▶ **Solution.** $\mathrm{Rank}(A) = \dim \mathrm{Col}(A) = 3.$ ◀

   (c) What is the dimension of the nullspace of A^T (Recall that A^T is the transpose of A.)

   ▶ **Solution.** $\mathrm{Rank}(A^T) = \mathrm{Rank}(A) = 3$ and thus

$$\dim(\text{Nullspace } (A^T)) = (\#\text{columns}) - \mathrm{Rank}(A^T) = 5 - 3 = 2.$$

◀

6. **[12 Points]** Complete the following definitions:

   (a) A set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of vectors in a vector space V is *linearly dependent* if ...

   ▶ **Solution.** there is a solution of the vector equation

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

   with at least one of the scalars $c_i \neq 0$. ◀

   (b) A *basis* of a vector space V is a set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of vectors such that ...

   ▶ **Solution.** S is linearly independent and $\mathrm{Span}(S) = V$. ◀

   (c) A vector $\mathbf{w}$ is a *linear combination* of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ if ...

   ▶ **Solution.** $\mathbf{w} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$ for some scalars $c_1, \ldots, c_p$. ◀