

Instructions. Answer each of the questions on your own paper, and be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper.

- [16 Points]** Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ be the basis of \mathbb{R}^2 with $\mathbf{v}_1 = (1, 2)$ and $\mathbf{v}_2 = (2, 3)$. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation for which $T(\mathbf{v}_1) = (3, 4)$ and $T(\mathbf{v}_2) = (5, 6)$. Find a formula for $T(x_1, x_2)$ and use it to compute $T(2, 5)$.
- [16 Points]** Find an invertible matrix P and diagonal matrix D such that $P^{-1}AP = D$, where $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.
- [12 Points]** Determine if the following functions are linear transformations. Give reasons.
 - $T : P_1 \rightarrow \mathbb{R}$ defined by $T(a + bx) = a + b$.
 - $S : P_1 \rightarrow \mathbb{R}$ defined by $S(a + bx) = ab$.
- [20 Points]** Determine the characteristic polynomial, all eigenvalues and all eigenvectors for the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & 1 & 1 \end{bmatrix}.$$

- [24 Points]** Let the linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be defined by

$$T(x, y, z) = (2x + y, x - z, 4x + y - 2z).$$

- Compute the matrix of T , denoted $[T]_{\mathcal{B}}$, with respect to the standard basis $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.
 - Find a basis for the kernel of T .
 - What is the dimension of the range of T ? Give a reason.
 - Find a basis for the range of T .
- [12 Points]** Let $T : \mathbb{R}^5 \rightarrow \mathbb{R}^3$ be a linear transformation. Let $r = \text{Rank}(T)$ and $\nu = \text{Nullity}(T)$. Determine whether statements (a) to (d) are:

[Always True] true for every linear transformation $T : \mathbb{R}^5 \rightarrow \mathbb{R}^3$

[Sometimes True] true for *some but not all* linear transformations $T : \mathbb{R}^5 \rightarrow \mathbb{R}^3$

[Always False] false for every linear transformation $T : \mathbb{R}^5 \rightarrow \mathbb{R}^3$.

$$(a) r \leq 3, \quad (b) r = 5 \text{ and } \nu = 0, \quad (c) \nu = 2 \text{ and } r = 3, \quad (d) \nu \geq 2.$$