Instructions. Answer each of the questions on your own paper, and be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper.

1. [**16 Points**] Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ be the basis of $\mathbb{R}^2$ with $\mathbf{v}_1 = (1, 2)$ and $\mathbf{v}_2 = (2, 3)$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation for which $T(\mathbf{v}_1) = (3, 4)$ and $T(\mathbf{v}_2) = (5, 6)$. Find a formula for $T(x_1, x_2)$ and use it to compute $T(2, 5)$.

   ▶ **Solution.** Write the vector $\mathbf{v} = (x_1, x_2) = c_1\mathbf{v_1} + c_2\mathbf{v_2} = c_1(1, 2) + c_1(2, 3)$, which gives the system of equations

$$\begin{aligned} x_1 &= c_1 + 2c_2 \\ x_2 &= 2c_1 + 3c_2. \end{aligned}$$

   Solve for c_1 and c_2 to get $c_1 = -3x_1 + 2x_2$ and $c_2 = 2x_1 - x_2$. Then T is a linear transformation so

$$\begin{aligned} T(x_1, x_2) &= T(c_1\mathbf{v_1} + c_2\mathbf{v_2}) \\ &= c_1T(\mathbf{v_1}) + c_2T(\mathbf{v_2}) \\ &= (-3x_1 + 2x_2)(3, 4) + (2x_1 - x_2)(5, 6) \\ &= (x_1 + x_2, 2x_2). \end{aligned}$$

   Thus, $T(2, 5) = (7, 10)$. ◀

2. [**16 Points**] Find an invertible matrix P and diagonal matrix D such that $P^{-1}AP = D$, where $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

   ▶ **Solution.** The matrix P has a basis of eigenvectors for A as columns, and D has the corresponding eigenvalues on the diagonal. To find the eigenvalues, compute the characteristic polynomial $c_A(\lambda) = \det(\lambda I - A) = \det \begin{bmatrix} \lambda - 1 & -3 \\ -3 & \lambda - 1 \end{bmatrix} = \lambda^2 - 2\lambda - 8 = (\lambda - 4)(\lambda + 2)$. The eigenvalues are the roots of the characteristic polynomial, and hence are $\lambda = 4$ and $\lambda = -2$. To find the eigenvalues of A with eigenvalue $\lambda = 4$, find the nullspace of the matrix $A = 4I = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}$. Thus, the eigenvectors with eigenvalue 4 are scalar multiples of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The eigenspace for the eigenvalue $\lambda = -2$ is the nullspace of the matrix $A - (-2)I = A + 2I = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$. Thus, the eigenvectors with eigenvalue -2 are scalar multiples of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Thus, we may take $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$. Then, P is invertible since $\det P = 2 \neq 0$ and $AP = PD$ so $P^{-1}AP = D$. ◀

3. [**12 Points**] Determine if the following functions are linear transformations. Give reasons.

(a) $T : P_1 \to \mathbb{R}$ defined by $T(a + bx) = a + b$.

▶ **Solution.** This is a linear transformation. To verify this, note that $T((a + bx) + (c + dx)) = T((a + c) + (b + d)x) = (a + c) + (b + d) = (a + b) + (c + d) = T(a + bx) + T(c + dx)$ and $T(k(a + bx)) = T(ka + kbx) = ka + kb = k(a + b) = kT(a + bx)$. Thus, T preserves vector addition and scalar multiplication, so it is a linear transformation. ◀

(b) $S : P_1 \to \mathbb{R}$ defined by $S(a + bx) = ab$.

▶ **Solution.** This is not a linear transformation, since it does not preserve vector addition: $S(1) = S(1 + 0x) = 1 \cdot 0 = 0$ and $S(x) = S(0 + x) = 0 \cdot 1 = 0$ so $S(1) + S(x) = 0$ but $S(1 + x) = 1 \cdot 1 = 1 \neq 0 = S(1) + S(x)$. ◀

4. [**20 Points**] Determine the characteristic polynomial, all eigenvalues and all eigenvectors for the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & 1 & 1 \end{bmatrix}.$$

▶ **Solution.** First compute the characteristic polynomial

$$\begin{aligned} c_A(\lambda) = \det(\lambda I - A) &= \det \begin{bmatrix} \lambda - 1 & 0 & 0 \\ 0 & \lambda - 3 & 1 \\ 0 & -1 & \lambda - 1 \end{bmatrix} \\ &= (\lambda - 1)((\lambda - 3)(\lambda - 1) + 1) \\ l &= (\lambda - 1)(\lambda^2 - 4\lambda + 4) \\ &= (\lambda - 1)(\lambda - 2)^2. \end{aligned}$$

Thus, the eigenvalues are 1 and 2, with the eigenvalue 2 having algebraic multiplicity 2.

Now find the eigenvectors associated to each eigenvalue.

$\lambda = 1$: Determine the eigenspace for the eigenvalue $\lambda = 1$. This is the nullspace of $A - I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, which is all multiples of the vector $\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$\lambda = 2$: We need to find the nullspace of the matrix $A - 2I = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$, which is all multiples of the vector $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. ◀

5. [**24 Points**] Let the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(x,\, y,\, z) = (2x + y,\, x - z,\, 4x + y - 2z).$$

(a) Compute the matrix of T, denoted $[T]_{\mathcal{B}}$, with respect to the standard basis $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

▶ **Solution.**

$$[T]_{\mathcal{B}} = \begin{bmatrix}[T(\mathbf{e}_1)]_{\mathcal{B}} & [T(\mathbf{e}_2)]_{\mathcal{B}} & [T(\mathbf{e}_3)]_{\mathcal{B}}\end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & -1 \\ 4 & 1 & -2 \end{bmatrix}.$$

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(b) Find a basis for the kernel of T.

▶ **Solution.** $\mathrm{Ker}(T) = \mathrm{Null}([T]_{\mathcal{B}})$. To compute the nullspace, row reduce the matrix to get

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

This is a rank 2 matrix so the nullspace has dimension 1. A basis is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, or in the standard form as an element of $\mathbb{R}^3$, we can write this is $\mathbf{v_1} = (1, -2, 1)$. ◀

(c) What is the dimension of the range of T? Give a reason.

▶ **Solution.** From the dimension theorem for linear transformations, $\dim(R(T)) + \dim \mathrm{Ker}(T) = 3$ so $\dim(R(T)) = 2$. ◀

(d) Find a basis for the range of T.

▶ **Solution.** The range of T is the column space of the matrix $[T]_{\mathcal{B}}$. Thus a basis consists of the first two columns of $[T]_{\mathcal{B}}$ (since these columns are linearly independent). In the standard form as elements of $\mathbb{R}^3$, the basis of the range of T is

$$S = \{(2, 1, 4), (1, 0, 1)\}.$$

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6. **[12 Points]** Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be a linear transformation. Let $r = \mathrm{Rank}(T)$ and $\nu = \mathrm{Nullity}(T)$. Determine whether statements (a) to (d) are:

[Always True] true for every linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$

[Sometimes True] true for *some but not all* linear transformations $T : \mathbb{R}^5 \to \mathbb{R}^3$

[Always False] false for every linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$.

(a) $r \leq 3$, (b) $r = 5$ and $\nu = 0$, (c) $\nu = 2$ and $r = 3$, (d) $\nu \geq 2$.

▶ **Solution.** (a) **Always true.**

(b) **Always False.**

(c) **Sometimes True.**

(d) **Always True.**

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