Instructions. Answer each of the questions on your own paper, and be sure to show your work so that partial credit can be adequately assessed. Put your name on each page of your paper.

1. [15 Points] Use induction to prove that for every integer \( n \geq 1 \),

\[
\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}.
\]

\[\textbf{Solution.}\] For \( n \in \mathbb{P} \), let \( P(n) \) be the statement:

\[
\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1}.
\]

For \( n = 1 \), \( P(1) \) is the statement:

\[
\frac{1}{1 \cdot 2} = \frac{1}{2},
\]

which is clearly a true statement. Thus \( P(1) \) is true, and the base step for induction is verified.

Now assume that \( P(m) \) is true. This means that

\[
\sum_{k=1}^{m} \frac{1}{k(k+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{m \cdot (m+1)} = \frac{m}{m+1}.
\]

Consider the left hand side of the statement \( P(n) \) for \( n = m + 1 \). That is:

\[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{m \cdot (m+1)} + \frac{1}{(m+1) \cdot (m+2)}.
\]

Because of the assumption that \( P(m) \) is true, \((*)\) can be written as

\[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{m \cdot (m+1)} + \frac{1}{(m+1) \cdot (m+2)} = \frac{m}{m+1} + \frac{1}{(m+1) \cdot (m+2)} = \frac{m(m+2) + 1}{(m+1)(m+2)} = \frac{(m+1)^2}{(m+1)(m+2)}.
\]

This is the statement that \( P(m+1) \) is true, provided \( P(m) \) is true, and by the principle of induction, we conclude that \( P(n) \) is a true statement for all \( n \in \mathbb{P} \). ▶

2. [14 Points]

(a) Compute the greatest common divisor \( d = (96, 87) \) of the integers 96 and 87 using the Euclidean Algorithm, and write \( d \) in the form \( d = 96 \cdot s + 87 \cdot t \).
Solution. We will use the matrix version of the Euclidean Algorithm:

\[
\begin{bmatrix}
1 & 0 & 96 \\
0 & 1 & 87 \\
\end{bmatrix} \sim \begin{bmatrix}
1 & -1 & 9 \\
0 & 1 & 87 \\
\end{bmatrix} \sim \begin{bmatrix}
1 & -1 & 9 \\
-9 & 10 & 6 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
10 & -11 & 3 \\
-9 & 10 & 6 \\
\end{bmatrix} \sim \begin{bmatrix}
10 & -11 & 3 \\
-29 & 32 & 0 \\
\end{bmatrix}
\]

Hence \( d = (96, 87) = 3 = 10 \cdot 96 - 11 \cdot 87. \)

(b) Compute the least common multiple \( m = [96, 87]. \)

Solution. \( m = [96, 87] = 96 \cdot 87 (96, 87) = 96 \cdot 87 \cdot 3 = 32 \cdot 87 = 2784. \)

3. [12 points]

(a) Find the prime factorization of 1260.

Solution. \( 1260 = 2^2 \cdot 3^2 \cdot 5 \cdot 7. \)

(b) Find positive integers \( m \) and \( n \) satisfying the following three conditions: (1) \( \gcd(m, n) = 3 \), (2) \( mn = 1260 \) and (3) the remainder when \( m \) is divided by \( n \) is 18.

Hint: Using part (a), make a table of all pairs \( m \) and \( n \) satisfying i. and ii. and see if this table contains a pair \( m \) and \( n \) satisfying iii.

Solution. \( \gcd(m, n) = 3 \) means that \( m = 3s \) and \( n = 3t \) where \( \gcd(s, t) = 1. \) Since \( mn = 1260 \) this means that \( st = 2^2 \cdot 5 \cdot 7 \) and \( s \) and \( t \) have no prime factors in common. Thus, the possible pairs \( (s, t) \) are (we only list those with \( s \geq t \), which gives \( m \geq n) \): (140, 1), (35, 4), (28, 5), and (20, 7). Thus, the pairs \( (m, n) \) are (420, 3), (105, 12), (84, 14), and (60, 21). Since \( 60 = 2 \cdot 21 + 18 \), the pair \( (m, n) = (60, 21) \) is the required pair that satisfies all three conditions.

4. [15 Points] This problem concerns arithmetic modulo 15. All answers should only involve expressions of the form \( [a]_{15} \) with \( a \) an integer satisfying \( 0 \leq a < 15. \)

(a) Compute \([7]_{15} - [4]_{15}. \) Answer: \([3]_{15}. \)

(b) Compute \([7]_{15}[4]_{15}. \) Answer: \([28]_{15} = [13]_{15}. \)

(c) Compute \([7]^{-1}_{15}. \) Answer: Since \( 15 - 2 \cdot 7 = 1, \) it follows that \([7]^{-1}_{15} = [-2]_{15} = [13]_{15}. \)

(d) List the invertible elements of \( \mathbb{Z}_{15}. \) (Recall that invertible means multiplicatively invertible.)

Solution. These are the elements \([a]_{15} \) where \( \gcd(a, 15) = 1. \) Thus, the invertible elements are:

(e) List the zero divisors of \( \mathbb{Z}_{15} \).

**Solution.** These are the remaining nonzero elements of \( \mathbb{Z}_{15} \):
\[
\]

5. [15 Points]

(a) State Euler’s Theorem concerning powers of \( a \) modulo \( n \) precisely. Be sure to carefully state the requisite hypotheses. Recall that \( \varphi(n) \) denotes the Euler \( \varphi \)-function applied to \( n \).

**Solution.** See Page 68 of the text.

(b) Compute \( \varphi(27) \), \( \varphi(29) \), and \( \varphi(72) \).

**Solution.**
\[
\varphi(27) = \varphi(3^3) = 3^3 - 3^2 = 27 - 9 = 18,
\]
\[
\varphi(29) = 29 - 1 = 28 \text{ since 29 is prime},
\]
\[
\varphi(72) = \varphi(2^3 \cdot 3^2) = \varphi(2^3)\varphi(3^2) = (2^3 - 2^2)(3^2 - 3^1) = 4 \cdot 6 = 24.
\]

(c) Compute \( 2^{63} \mod 29 \).

**Solution.** Writing 63 = 2\cdot28+7 and applying Fermat’s Theorem with \( p = 29 \) gives \( 2^{28} \equiv 1 \mod 29 \). Hence
\[
2^{63} \equiv 2^{2\cdot28+7} \equiv (2^{28})^2 \cdot 2^7 \equiv 2^7 \mod 29
\]
\[
\equiv 2^5 \cdot 2^2 \equiv 32 \cdot 2^2 \mod 29
\]
\[
\equiv 3 \cdot 4 \equiv 12 \mod 29.
\]

6. [14 Points] Solve the following linear congruences:

(a) \( 5x \equiv 7 \mod 31 \);

**Solution.** From the equation \( 31 - 6 \cdot 6 = 1 \) we see that \( [5]_{31}^{-1} = [-6]_{31} = [25]_{31} \). Hence, multiplying the congruence by \(-6\) gives
\[
x \equiv (-6)5x \equiv (-6) \cdot 7 \equiv -42 \equiv 20 \mod 31.
\]

(b) \( 2x \equiv 19 \mod 2008 \).

**Solution.** Since \( \gcd(2, 2008) = 2 \) and \( 2 \nmid 19 \), this congruence has no solutions.
7. [15 Points] Find the smallest positive integer \( x \) that has a remainder 4 when divided by 7 and a remainder 5 when divided by 25.

▶ Solution. We are looking for the smallest positive solution of the pair of simultaneous congruences

\[
\begin{align*}
    x &\equiv 4 \pmod{7} \\
    x &\equiv 5 \pmod{25}
\end{align*}
\]

This can be solved by using the Chinese Remainder Theorem as follows: Write \( 2 \cdot 25 - 7 \cdot 7 = 1 \). Then

\[
x \equiv 4 \cdot (2 \cdot 25) + 5 \cdot (-7) \cdot 7 \equiv -45 \pmod{7 \cdot 25}.
\]

The smallest such positive number is \( x = -45 + 175 = 130 \).