Instructions. Answer each of the questions on your own paper, and be sure to show your work so that partial credit can be adequately assessed. Credit will not be given for answers (even correct ones) without supporting work. Put your name on each page of your paper.

1. [10 Points] What is the minimum number of students in a class to guarantee that at least 5 of them were born in the same month (regardless of the year of their birth)?

2. [15 Points]
   (a) Find the greatest common divisor \( d = (7605, 5733) \) of 7605 and 5733, using the Euclidean Algorithm.
   (b) Write \( d = (7605, 5733) \) in the form \( d = s \cdot 7605 + t \cdot 5733 \).
   (c) Find the least common multiple \([7605, 5733]\). (The formula \([a, b]\)(a, b) = ab may be useful.)

3. [15 Points]
   (a) What is the relationship between \( a \) and \( n \) which guarantees that \( a \) has a multiplicative inverse in \( \mathbb{Z}_n \)? (Just state the condition. It is not necessary to verify it.)
   (b) Find the multiplicative inverse of 13 in \( \mathbb{Z}_{225} \).

4. [15 Points] In the group \( S_{10} \), let \( \alpha = (1 \ 3 \ 5 \ 7 \ 9) \), \( \beta = (1 \ 2 \ 6) \), and \( \gamma = (1 \ 2 \ 5 \ 3) \).
   (a) If \( \sigma = \alpha \beta \gamma \), write \( \sigma \) as a product of disjoint cycles, and use this to find its order and its inverse.
   (b) Is \( \sigma \) even or odd?
   (c) Solve the group equation \( x \beta = \gamma \) for \( x \). Express \( x \) as a product of disjoint cycles.

5. [15 Points] Let \( G \) be a group and let \( a \) and \( b \) be elements of \( G \). Show by induction that \( (aba^{-1})^n = ab^n a^{-1} \) for all natural numbers \( n \in \mathbb{P} \).

6. [15 Points]
   (a) State Lagrange’s Theorem.
   (b) Suppose that a group \( G \) has subgroups of order 9 and 12. If \( |G| < 150 \), what are the possibilities for \( |G| \)?

7. [15 Points] The elements of \( \mathbb{Z}_{16} \) which have a multiplicative inverse form a group \( G \) with the following multiplication table:

\[
\begin{array}{cccccccc}
\cdot & 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 \\
1 & 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 \\
3 & 3 & 9 & 15 & 5 & 11 & 1 & 7 & 13 \\
5 & 5 & 15 & 9 & 3 & 13 & 7 & 1 & 11 \\
7 & 7 & 5 & 3 & 1 & 15 & 13 & 11 & 9 \\
9 & 9 & 11 & 13 & 15 & 1 & 3 & 5 & 7 \\
11 & 11 & 1 & 7 & 13 & 3 & 9 & 15 & 5 \\
13 & 13 & 7 & 11 & 5 & 15 & 9 & 3 & 15 \\
15 & 15 & 13 & 11 & 9 & 7 & 5 & 3 & 1 \\
\end{array}
\]
(a) Let $H = \{1, 7\}$. Explain why $H$ is a subgroup of $G$.
(b) List all of the left cosets of $H$.
(c) Write the multiplication table for the quotient group $G/H$.
(d) Is $G/H$ a cyclic group? Explain.

8. [20 Points] Let $C$ be the binary linear code with generator matrix

$$
\begin{bmatrix}
1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1
\end{bmatrix}
$$

and let $\tilde{C}$ be the ternary linear code with the same generator matrix $G$.

(a) List all of the codewords of $C$.
(b) How many codewords does $\tilde{C}$ have? (Do not list them.)
(c) Write parity check matrices $H$ for $C$ and $\tilde{H}$ for $\tilde{C}$.
(d) Is the word $\hat{x} = 2120102$ a code word for $\tilde{C}$? Explain.
(e) What are the parameters $(n, k, d)$ for the code $C$? Be sure to give reasons for your answers.
(f) How many errors can $C$ detect and how many errors can $C$ correct?

9. [15 Points] Let $C$ be the $(4, 2, 3)$ ternary linear code with parity check matrix

$$
H = \begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 2
\end{bmatrix}.
$$

That is, $C$ is the Hamming code Ham$(2, 3)$.

(a) List all of the words $\hat{x} \in (Z_3)^4$ such that $d(\hat{x}, \hat{c}) \leq 1$, where $\hat{c} = 1201$.
(b) Decode each of the following words using syndrome decoding.
   i. 2021 
   ii. 1221 
   iii. 1011

10. [15 Points] A wheel is divided evenly into five different compartments. Each compartment can be painted purple or gold. The back of the wheel is black. How many different such color wheels are there?