The set of even permutations in S_n is written A_n. It can be easily shown that A_n is a subgroup of S_n; A_n is called the *alternating group of degree n*.

EXERCISES 5.1

1. Find the number of distinct cycles of length r in S_n.
2. Write the cycle structure classification table for S_5.
3. Write the cycle structure classification table for S_6.
4. Prove that the relation $\sim$ (of conjugate elements) in a group is an equivalence relation.
5. Show that the permutations

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 5 & 1 & 6 & 4 & 2 & 3 \end{pmatrix}$$

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 1 & 6 & 2 & 3 & 7 & 5 \end{pmatrix}$$

   have the same cycle structure. Find σ such that $\beta = \sigma\alpha\sigma^{-1}$.
6. Let k_n denote the number of permutations in S_n that do not fix any element. Prove that

$$k_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{n-1}}{n!}\right)$$

   Hence show that k_n is equal to the integer closest to $\frac{n!}{e}$.
7. Let X be an infinite set. Let N denote the set of those permutations of X that move a finite number of elements in X. Prove that N is a normal subgroup of S_X.
8. Show that for every $n > 1$, A_n is a normal subgroup of S_n and $[S_n : A_n] = 2$.
9. Show that for every $n > 2$, the center of the group S_n is trivial.

5.2 GROUPS OF SYMMETRIES

In this section we consider the application of group theory to study the symmetries of a geometrical figure (in a plane or three-dimensional space).

Let $d(x, y)$ denote the distance between the points x and y.

Definition 5.2.1 Let X be a set of points. A bijective mapping $\sigma : X \to X$ is called a *symmetry* of X if

$$d(\sigma(x), \sigma(y)) = d(x, y) \qquad \text{for all } x, y \in X$$

In other words, a symmetry of a set of points X is a permutation of X that preserves the distance between every two points in X.

We write $Sym(X)$ to denote the set of all symmetries of X. Obviously, $Sym(X)$ is a subset of S_X.

THEOREM 5.2.2 Let X be a set of points. Then $Sym(X)$ is a subgroup of the symmetric group S_X.

Proof: The identity permutation $e \in S_X$ is obviously a symmetry; hence $e \in Sym(X)$. Let $\alpha, \beta \in Sym(X)$. Then, for all $x, y \in X$,

$$\begin{aligned} d(\alpha^{-1}\beta(x), \alpha^{-1}\beta(y)) &= d(\alpha(\alpha^{-1}\beta(x)), \alpha(\alpha^{-1}\beta(y))) \\ &= d(\beta(x), \beta(y)) \\ &= d(x, y) \end{aligned}$$

Hence $\alpha^{-1}\beta \in Sym(X)$. This proves that $Sym(X)$ is a subgroup of S_X. ∎

The group $Sym(X)$ is called the *group of symmetries* (or *symmetry group*) of X. Note that the symmetric group S_X is defined for any set X, but $Sym(X)$ is defined for only a set X of points in which the distance between every two points is given.

Consider the symmetries of a polygon P. (By P we mean the set of points constituting the polygon.) Let V be the set of vertices of the polygon. It is clear from geometrical consideration that any symmetry σ of the polygon must map a vertex to a vertex. Thus σ determines a symmetry $\bar{\sigma}$ of the set V. Conversely, given any symmetry $\bar{\sigma}$ of V, it determines uniquely a symmetry σ of the polygon that coincides with $\bar{\sigma}$ on the vertices. Hence we can identify the symmetries of the polygon with the symmetries of the set of its vertices. In other words, speaking more formally, the group of symmetries of P is isomorphic with the group of symmetries of V.

Let us now consider a regular polygon of n sides ($n \geq 3$). Let us label the vertices in counterclockwise order as $1, 2, \ldots, n$. Consider any symmetry σ of the set of vertices. Suppose σ maps vertex 1 to vertex i. Then σ must take vertex 2 to a vertex adjacent to i—that is, either $i+1$ or $i-1$. Once $\sigma(1)$ and $\sigma(2)$ are fixed, the mapping σ is completely determined by the fact that it preserves the distance between every two points. So if σ maps 2 to $i+1$, then it must map $3, 4, \ldots$ to $i+2, i+3, \ldots$, respectively. On the other hand, if σ maps 2 to $i-1$, then it must map $3, 4, \ldots$ to $i-2, i-3, \ldots$, respectively. Thus there are exactly two symmetries, σ_i and τ_i, that take vertex 1 to i. These are given by

$$\sigma_i = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ i & i+1 & i+2 & \ldots & i-1 \end{pmatrix}$$

$$\tau_i = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ i & i-1 & i-2 & \ldots & i+1 \end{pmatrix}$$

Thus we see that a regular polygon of n sides has in all $2n$ symmetries, σ_i, τ_i, $i = 1, \ldots, n$.

Note that the vertex after n is 1, so $i+1=1$ when $i=n$. Therefore, in the above representation for σ_i, τ_i, $+$ is to be understood as addition modulo n.

The mapping σ_i preserves the cyclic order of the vertices, but τ_i reverses the cyclic order. Geometrically, σ_i represents a rotation of the polygon about its center through an angle $2\pi(i-1)/n$, and τ_i represents a reflection in the diameter lying midway between vertices 1 and i. (By a rotation, we mean a rotation of the polygon in its own plane. A reflection in a diameter is equivalent to a rotation about the diameter through an angle π, but this rotation takes place in the third dimension and not in the plane of the polygon.) It is obvious that σ_1 is the identity permutation, and τ_1 represents reflection in the diameter through vertex 1. The identity permutation is equivalent to a rotation through an angle 2π.

The $2n$ symmetries $\sigma_i, \tau_i, i=1,\ldots,n$, can be expressed in terms of two basic symmetries. We write $\alpha=\sigma_2$, $\beta=\tau_1$, so

$$\alpha=\begin{pmatrix}1 & 2 & \ldots & n\\ 2 & 3 & \ldots & 1\end{pmatrix}, \quad \beta=\begin{pmatrix}1 & 2 & \ldots & n\\ 1 & n & \ldots & 2\end{pmatrix}$$

Geometrically, α represents a rotation through angle $2\pi/n$ and moves each vertex i to $i+1$. For any integer $m=1,\ldots,n$, α^{m-1} represents a rotation through angle $2\pi(m-1)/n$; hence $\alpha^{m-1}=\sigma_m$. Further $\alpha^{m-1}\beta(1)=\alpha^{m-1}(1)=m$ and $\alpha^{m-1}\beta(2)=\alpha^{m-1}(n)=m-1$. Since a symmetry is determined uniquely by its effect on vertices 1 and 2, it follows that $\alpha^{m-1}\beta=\tau_m$. Thus the $2n$ symmetries are given by $\alpha^{m-1}, \alpha^{m-1}\beta$, $m=1,\ldots,n$.

It is clear that $\alpha^m=e$ and $\beta^2=e$. Further, consider $\beta\alpha$: $\beta\alpha(1)=\beta(2)=n$ and $\beta\alpha(2)=\beta(3)=n-1$. Hence $\beta\alpha=\tau_n=\alpha^{n-1}\beta$. Thus we have proved the following result.

THEOREM 5.2.3 The group G of symmetries of a regular polygon of n sides is given by

$$G=\{e,\alpha,\ldots,\alpha^{n-1},\beta,\alpha\beta,\ldots,\alpha^{n-1}\beta\}$$

where α represents a rotation through an angle $2\pi/n$, and β represents reflection in a diameter through a vertex. Moreover, the following relations hold in the group G:

$$\alpha^n=e, \quad \beta^2=e, \quad \beta\alpha=\alpha^{n-1}\beta$$

Any group of $2n$ elements that has the same structure as the group G above is called a *dihedral group* of degree n and denoted by D_n. That is, we have the following definition.

Definition 5.2.4 A *dihedral group* of degree n, written D_n, is a group of order $2n$ given by

$$D_n=\{e,a,\ldots,a^{n-1},b,ab,\ldots,a^{n-1}b\}$$

with the following defining relations:

$$a^n = e, \quad b^2 = e, \quad ba = a^{n-1}b$$

We have thus shown that the group of symmetries of a regular polygon of n sides ($n \geq 3$) is a dihedral group of degree n. We shall shortly explain the geometrical interpretation of the dihedral groups D_1 and D_2 as groups of symmetries. For the present, let us observe that $D_1 = \{e, b\}$, with $b^2 = e$, is a cyclic group of order 2. Further, $D_2 = \{e, a, b, ab\}$ has the defining relations $a^2 = e$, $b^2 = e$, and $ba = ab$. Hence D_2 is identical with a Klein's 4-group.

If we interpret the elements of the dihedral group D_n, $n > 2$, as permutations of the set $\{1, \ldots, n\}$ of vertices of a regular polygon, then D_n is a subgroup of the symmetric group S_n. In particular, D_3 has six elements and hence $D_3 = S_3$. For $n > 3$, D_n is a proper subgroup of S_n.

An equilateral triangle is a regular polygon of three sides. Hence its group of symmetries is D_3. In this case, we can also arrive at this result directly as follows: The distance between every pair of vertices of an equilateral triangle is the same, and hence every permutation of the vertices is a symmetry. Therefore the group of symmetries of an equilateral triangle is S_3, which, as noted above, is the same as D_3.

Consider now an isosceles (but not equilateral) triangle. It has only one symmetry β in addition to the identity permutation—namely, the one given by reflection in the median bisecting the angle between the two equal sides. So the group G of symmetries of an isosceles triangle is given by $G = \{e, \beta\}$, with $\beta^2 = e$. As noted above, G is a dihedral group of degree 1.

Consider next the symmetries of a rectangle (other than a square). It is easily seen that there are only three symmetries α, β, γ in addition to e. Geometrically, these represent a rotation through an angle π and reflections in the lines through the center and parallel to the sides of the rectangle. Labeling the vertices as 1, 2, 3, 4 in order, we can write these symmetries as permutations of the vertices as follows:

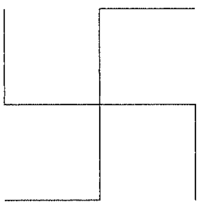
$$\alpha = (1\,3)(2\,4), \quad \beta = (1\,2)(3\,4), \quad \gamma = (1\,4)(2\,3)$$

It is easily verified that $\alpha\beta = \gamma = \beta\alpha$. Hence the group of symmetries of a rectangle is given by $G = \{e, \alpha, \beta, \alpha\beta\}$, with the defining relations $\alpha^2 = e$, $\beta^2 = e$, $\beta\alpha = \alpha\beta$. So G is a dihedral group of degree 2.

We can summarize the results proved above as follows: The group of symmetries of an isosceles triangle is D_1. The group of symmetries of a rectangle is D_2. For any $n > 2$, the group of symmetries of a regular polygon of n sides is D_n.

The dihedral group D_n has a subgroup $C_n = \{e, a, \ldots, a^{n-1}\}$ that, geometrically, consists of all rotational symmetries of the polygon.

The symmetry group of a geometric figure may be infinite. For example, a circle has infinitely many symmetries. It can be shown that if the symmetry group G of a plane figure is finite, then G is either D_n or C_n for some n. For example, the symmetry group of this figure is C_4.



Let us now consider the symmetries of three-dimensional geometric objects. Consider first a regular tetrahedron. Let the vertices be labeled as 1, 2, 3, 4. As in the case of an equilateral triangle, the distance between every two vertices of a regular tetrahedron is the same. Hence every permutation of the vertices is a symmetry. Therefore the symmetry group of a regular tetrahedron is S_4. How many of the 24 permutations in S_4 are rotations? It is clear that by a suitable rotation we can take vertex 1 to any vertex $i = 1, 2, 3, 4$. Having done that, we can rotate the tetrahedron about an axis through the new position of vertex 1 through angles 0, $2\pi/3$, and $4\pi/3$ to obtain three symmetries. Thus there are in all $4 \times 3 = 12$ rotational symmetries. They form a subgroup of the group of all symmetries of the tetradedron.

The following table gives the 12 rotational symmetries of a regular tetrahedron as permutations of the vertices and their geometric description as rotations. The edge $i - j$ denotes the edge joining vertices i and j.

Permutation	**Axis and Angle of Rotation**
$(1) = e$	any axis, rotation through angle 2π
$(2\,3\,4)$, $(2\,4\,3)$	axis through vertex 1, angles $2\pi/3$ and $4\pi/3$
$(1\,3\,4)$, $(1\,4\,3)$	axis through vertex 2, angles $2\pi/3$ and $4\pi/3$
$(1\,2\,4)$, $(1\,4\,2)$	axis through vertex 3, angles $2\pi/3$ and $4\pi/3$
$(1\,2\,3)$, $(1\,3\,2)$	axis through vertex 4, angles $2\pi/3$ and $4\pi/3$
$(1\,2)(3\,4)$	axis through middle points of edges 1-2 and 3-4, angle π
$(1\,3)(2\,4)$	axis through middle points of edges 1-3 and 2-4, angle π
$(1\,4)(2\,3)$	axis through middle points of edges 1-4 and 2-3, angle π

Let us now consider the symmetries of a cube. Let the vertices of the cube be labeled $1, 2, \ldots, 8$ such that vertices 2, 3, and 4 are adjacent to vertex 1. Let $\sigma \in S_8$ be a symmetry. Suppose σ takes 1 to i. Then σ must take 2, 3, and 4 to the three vertices adjacent to i, which can be done in 6 ways. Hence there are $8 \times 6 = 48$ symmetries in all. Of these, 24 are rotational symmetries. The vertex 1 can be taken to any vertex i by a rotation, and then we can rotate the cube around the diameter through the new position of vertex 1 to obtain three symmetries.

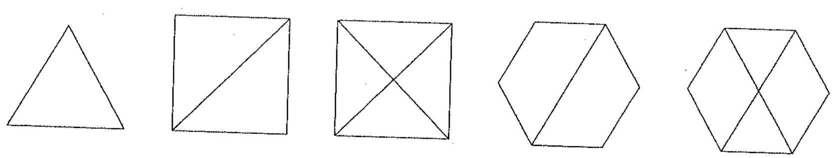
We can arrive at the same result by considering the symmetries of the cube as permutations of its six faces. By a rotation, face 1 can be taken to face i $(i = 1, \ldots 6)$. Having done that, we can rotate the cube around the diameter perpendicular to the new position of the face 1 through angles $\pi/2, \pi$, and $3\pi/2$ and obtain three symmetries. Hence there are $6 \times 4 = 24$ rotational symmetries.

The following table gives a geometric description of the various types of rotational symmetries of a cube and their numbers.

Axis and Angle of Rotation	Number
any axis, angle 2π	1
axis through opposite vertices, angles $2\pi/3$ and $4\pi/3$	$4 \times 2 = 8$
axis through centers of opposite faces, angles $\pi/2$, π, and $\frac{3\pi}{2}$	$3 \times 3 = 9$
axis through middle points of opposite edges, angle π	6

EXERCISES 5.2

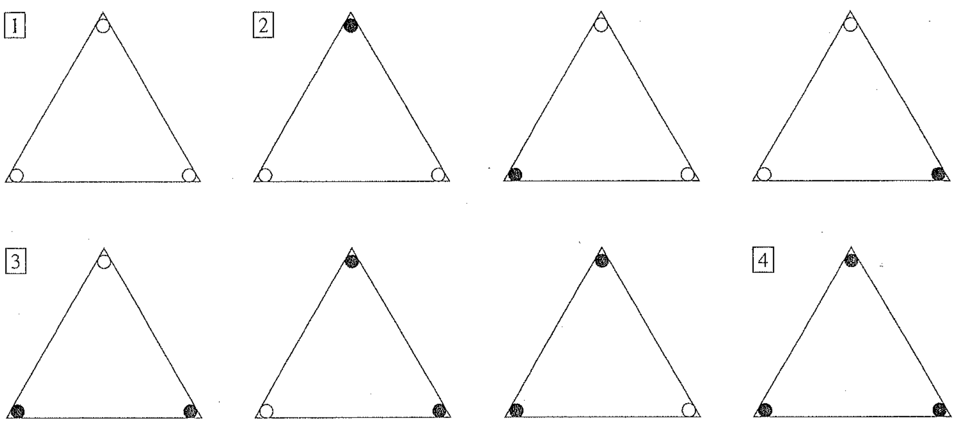
1. Find the symmetry groups of the following figures.



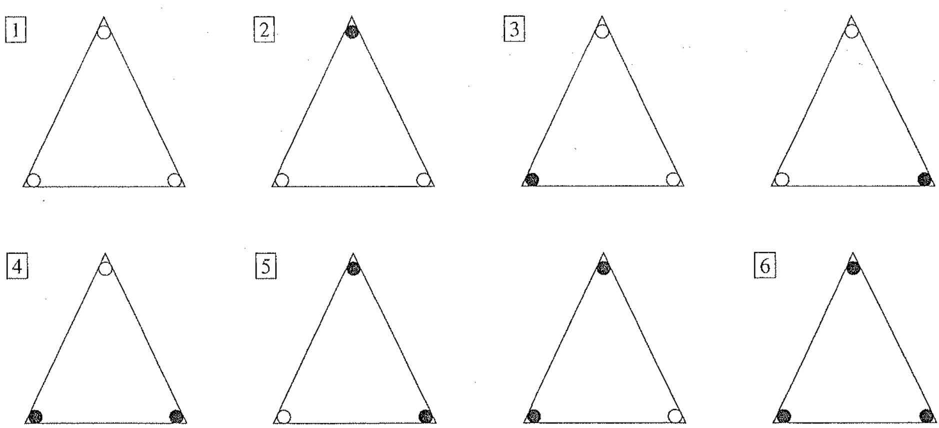
2. Find the symmetry groups of the letters of the alphabet: A, B, . . . , Z.
3. Find the symmetry groups of the conic sections ellipse, parabola, and hyperbola.
4. Find the symmetry groups of the following curves:
   (a) $y^2 = x(1 - x^2)$
   (b) $y^2 = x^2(1 - x^2)$
   (c) $r = 1 + \cos\theta$
   (d) $r = \sin 2\theta$
   (e) $r = \sin 3\theta$
5. Write the rotational symmetries of a cube as permutations of the vertices.
6. Write the rotational symmetries of a cube as permutations of the edges.
7. Show that the rotational symmetry group of a regular tetrahedron is isomorphic with A_4.
8. Show that the rotational symmetry group of a cube is isomorphic with S_4.
9. Show that the center of the group D_n is of order 1 or 2 according to whether n is odd or even.

5.3 COLORINGS AND COLOR PATTERNS

In this chapter we consider problems of the following type: Suppose we color each vertex of an equilateral triangle white or black. Then there are $2 \times 2 \times 2 = 8$ ways in which the three vertices can be colored. Let us refer to them as *color assignments* or *colorings*. We say that two color assignments are *equivalent* (or have the same *pattern*) if one of them can be obtained from the other by rotating the triangle through an appropriate angle or flipping it over. The second operation—namely, flipping over—is equivalent to reflection in some mirror line. We then find that the eight color assignments fall into four distinct patterns, as shown here.

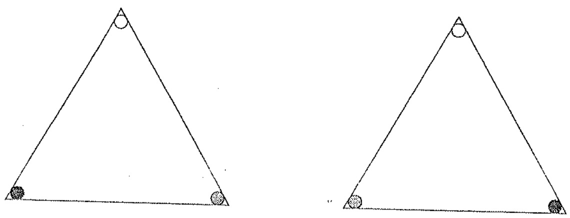


If, instead of an equilateral triangle, we consider an isosceles triangle, then we find that the eight color assignments fall into six distinct patterns.



Finally, if we consider a triangle whose sides are all of unequal lengths, then no two colorings are equivalent, and hence all eight colorings are distinct patterns.

Now suppose we make the criterion for equivalent colorings stricter by stipulating that two colorings are equivalent if one can be obtained from the other by a rotation only (and not reflection). Then we find that no two colorings of an isosceles triangle are equivalent, so we have eight different patterns. In the case of an equilateral triangle, however, this makes no difference, and we still have only four patterns. But now suppose we use three colors, instead of only two. Then the two colorings of an equilateral triangle shown below are equivalent if we allow reflection, but they are not equivalent if we use the stricter criterion of allowing only rotation.



We now formulate the general problem that we intend to consider. Let S be a finite set whose elements are specified points or parts of some given geometric figure. Suppose we assign to each element in S a color out of some given set of m colors. The total number of ways in which such color assignments can be made is $m \times m \times \cdots \times m = m^n$, where n is the number of elements in S. The problem is to find the number of distinct patterns in which these m^n color assignments fall. For recognizing distinct patterns, we may use either the weaker criterion of allowing both rotations and reflections or the stricter criterion of allowing only rotations. It is obvious that the number of distinct patterns under the weaker criterion is less than or equal to the number under the stricter criterion.

It is clear from the examples discussed above that the number of distinct patterns into which the m^n color assignments fall depends not only on the numbers m and n but also on the symmetry properties of the underlying geometric figure. The greater the symmetry possessed by the figure, the larger the number of equivalent pairs of colorings and therefore the smaller the number of distinct patterns.

5.4 ACTION OF A GROUP ON A SET

Let X be a nonempty set, and let G be a permutation group on X; that is, G is a subgroup of the symmetric group S_X. So each element of G is a permutation of the set X. Hence, for all $g \in G$, $x \in X$, $g(x)$ is again an element of X. Moreover, since the group operation in G is composition of mappings, we have the following two properties for all $x \in X$:

1. $e(x) = x$, where e is the identity in G.
2. $(gh)(x) = g(h(x))$ for all $g, h \in G$.

These two properties motivate the general concept of action of a group on a set.

Definition 5.4.1 Let G be a group, and let X be a nonempty set. A mapping $*: G \times X \to X$, with $*(g, x)$ written $g * x$, is called an *action* of G on X if the following conditions hold for all $x \in X$:

(a) $e * x = x$, where e is the identity in G.
(b) $(gh) * x = g * (h * x)$ for all $g, h \in G$.

If there is an action of a group G on a set X, we say that G *acts* on X and call X a G-*set.*

EXAMPLES

1. Given any nonempty set X, let G be a subgroup of the symmetric group S_X. For any $g \in G$ and $x \in X$, we define $g * x = g(x)$. Then it follows from conditions (a) and (b) in Definition 5.4.1 that $*$ is an action of G on X. We say in this case that G acts *naturally* on X. In particular, if G is the group of symmetries of a set X of points in space, then G acts naturally on X.
2. Given any group G, let $X = G$. We define $g * x = gx$ (the product of g and x in the group G). Then $*$ is an action of G on X. We say in this case that G acts on itself by *left translation.* Further, if H is a subgroup of G, then G acts on the quotient set G/H by left translation on taking $g * (aH) = (ga)H$.
3. Given any group G, again take $X = G$. We define $g * x = gxg^{-1}$. Then $e * x = exe^{-1} = x$ and $(gh) * x = ghx(gh)^{-1} = g(hxh^{-1})g^{-1} = g * (h * x)$ for all $x \in X$ and $g, h \in G$. Hence G acts on X. We say in this case that G acts on itself by *conjugation.* Further, if N is a normal subgroup of G, then G acts on the quotient set G/N by conjugation on taking $g * (aH) = g(aH)g^{-1}$.

We saw in Example 1 that if G is a subgroup of S_X, then G acts naturally on X. Conversely, we can show that if a group G acts on a set X, then G determines a subgroup of S_X that is a homomorphic image of G.

Let G be a group acting on a set X. Given $g \in G$, we define the mapping $\sigma_g : X \to X$ by the rule $\sigma_g(x) = g * x$. Then σ_g is a permutation of X. Let $x, y \in X$. Then $\sigma_g(x) = \sigma_g(y) \Rightarrow g * x = g * y \Rightarrow g^{-1} * (g * x) = g^{-1} * (g * y) \Rightarrow (g^{-1}g) * x = (g^{-1}g) * y \Rightarrow e * x = e * y \Rightarrow x = y$. Hence σ_g is injective. Further, given $y \in X$, let $x = g^{-1} * y$. Then $\sigma_g(x) = g * x = g * (g^{-1} * y) = (g^{-1}g) * y = e * y = y$. Hence σ_g is surjective. This proves that σ_g is a permutation of X.

Consider now the mapping $\phi : G \to S_X$ given by $g \mapsto \sigma_g$. Let $g, h \in G$. Then for all $x \in X$, $\sigma_{gh}(x) = (gh) * x = g * (h * x) = \sigma_g(\sigma_h(x)) = (\sigma_g\sigma_h)(x)$. Hence $\phi(gh) = \phi(g)\phi(h)$. This proves that ϕ is a homomorphism. Hence Im ϕ is a subgroup of S_X and a homomorphic image of G. We write G_X to denote Im ϕ and refer to it as the permutation group on X *induced* by G.

Moreover, if the identity e in G is the only element such that $e * x = x$ for all $x \in X$, then $\ker \phi = \{e\}$, and so ϕ is an isomorphism and G is isomorphic with G_X. In this case we can identify each g in G with σ_g, and consider G as a subgroup of S_X.

For the sake of convenience, we shall henceforth write simply gx instead of $g * x$.

Let G be a group acting on a set X. For any $a \in X$, the *orbit* of a under G is defined to be the set

$$Orb(a) = \{ga \mid g \in G\}$$

THEOREM 5.4.2 Let G be a group acting on a set X. Then the orbits in X under G form a partition of X.

Proof: For any $x, y \in X$, let $x \sim y$ mean that $x = gy$ for some $g \in G$. We claim that $\sim$ is an equivalence relation. For any $x \in X$, $x = ex$ and hence $x \sim x$. If $x = gy$ for some $g \in G$, then $g^{-1}x = g^{-1}(gy) = (g^{-1}g)y = ey = y$. Hence $x \sim y \Rightarrow y \sim x$. If $x = gy$ and $y = hz$ for some $g, h \in G$, then $x = (gh)z$ and hence $x \sim y$, $y \sim z \Rightarrow x \sim z$. This proves that $\sim$ is an equivalence relation in X. For any $a \in X$, the equivalence class of a under $\sim$ is

$$Cl_{\sim}(a) = \{x \in X \mid x \sim a\} = \{ga \mid g \in G\} = Orb(a)$$

Hence the orbits in X under G are the equivalence classes under $\sim$ and therefore form a partition of X. ∎

The partition of X formed by the set of orbits under G is denoted by X/G and called the *orbit decomposition* of X under G.

Again, let G be a group acting on a set X. Let $g \in G$ and $x \in X$. If $gx = x$, we say g *fixes* x. Given $x \in X$, the set of all elements in G that fix x is called the *stabilizer* of x and written $Stab(x)$; that is,

$$Stab(x) = \{g \in G \mid gx = x\}$$

THEOREM 5.4.3 Let G be a group acting on a set X, and let $x \in X$. Then

(a) $Stab(x)$ is a subgroup of G.
(b) The index of the subgroup $Stab(x)$ in G is

$$(G : Stab(x)) = |Orb(x)|$$

Proof: **(a)** We write $S = Stab(x)$. Since $ex = x$, we have $e \in S$. If $g, h \in S$, then $(g^{-1}h)x = g^{-1}(hx) = g^{-1}x = g^{-1}(gx) = (g^{-1}g)x = ex = x$ and hence $g^{-1}h \in S$. This proves that S is a subgroup of G.

(b) As usual, let G/S denote the set of left cosets of S in G. We define

$$\phi : G/S \to Orb(x)$$

by the rule

$$\phi(gS) = gx$$

We claim that ϕ is a bijective mapping. If $gS = hS$, then $g^{-1}h \in S$; hence $g^{-1}hx = x$ and therefore $gx = g(g^{-1}hx) = hx$. This shows that ϕ is well defined. If $\phi(gS) = \phi(hS)$, then $gx = hx$ and hence $g^{-1}hx = x$, which implies that $g^{-1}h \in S$; therefore $gS = hS$. This proves that ϕ is injective. If $y \in Orb(x)$, then $y = gx = \phi(gS)$ for some $g \in G$. Hence ϕ is surjective and therefore bijective. Hence $(G : Stab(x)) = |G/S| = |Orb(x)|$. ∎

Now we prove the main theorem that enables us to count the number of color patterns. Given $g \in G$, the set of all elements in X that are fixed by g is called the *fixture* of g and written $Fix(g)$; that is,

$$Fix(g) = \{x \in X \mid gx = x\}$$

THEOREM 5.4.4 (Burnside theorem) Let G be a finite group acting on a finite set X. Then the number k of orbits in X under G is given by

$$k = \frac{1}{|G|} \sum_{g \in G} F(g)$$

where $F(g) = |\ Fix(g)|$ is the number of elements in X that are fixed by g.

Proof: We count in two ways the number of ordered pairs (g, x) in $G \times X$ such that g fixes x. We write

$$P = \{(g, x) \in G \times X \mid gx = x\}$$

If $gx = x$, then $g \in Stab(x)$ and $x \in Fix(g)$. Hence, given $x \in X$, the number of elements in G that fix x is equal to $|Stab(x)|$. On the other hand, given $g \in G$, the number of elements in X that are fixed by g is equal to $|Fix(g)|$. Hence

$$\sum_{x \in X} |Stab(x)| = |P| = \sum_{g \in G} |Fix(g)| \tag{1}$$

By Theorem 5.4.3,

$$|Orb(x)| = (G : Stab(x)) = \frac{|G|}{|Stab(x)|}$$

Hence

$$\sum_{x \in X} |Stab(x)| = |G| \sum_{x \in X} \frac{1}{|Orb(x)|} \tag{2}$$

Now for any orbit $T \in X/G$,

$$\sum_{x\in T} \frac{1}{|Orb(x)|} = \sum_{x\in T} \frac{1}{|T|} = |T| \frac{1}{|T|} = 1$$

Therefore, since the orbits in X form a partition of X,

$$\sum_{x\in X} \frac{1}{|Orb(x)|} = \sum_{T\in X/G} \sum_{x\in T} \frac{1}{|Orb(x)|} = \left|\frac{X}{G}\right| = k$$

Hence, using (1) and (2), we obtain

$$k = \sum_{x\in X} \frac{1}{|Orb(x)|} = \frac{1}{|G|} \sum_{x\in X} |Stab(x)| = \frac{1}{|G|} \sum_{g\in G} |Fix(g)|$$

This completes the proof. ∎

Burnside theorem can be paraphrased as follows: If a group G acts on a set X, then the number of orbits in X under G is equal to the average number of elements in X fixed by an element in G.

5.5 BURNSIDE THEOREM AND COLOR PATTERNS

We now take up the original problem that we posed earlier in this chapter: finding the number of patterns in the colorings of a given set of points. We shall see how Burnside theorem is used to obtain the solution.

Using our earlier notation, we let S be a set of n elements representing some specified points (or parts) of a given geometric figure. To each element in S we assign some color out of a given set of m colors. This can be done in m^n ways, which we refer to as *m-colorings* of S.

Let X denote the set of all m-colorings of S. Let G be a group of symmetries of the set S. The group G acts naturally on the set S. Therefore G also acts on the set X. Given $g \in G$ and $x \in X$, gx represents the color assignment obtained by performing on the coloring x the symmetry operation (rotation or reflection) represented by g. Two color assignments x and y are equivalent if and only if $y = gx$ for some g in G. Hence all color assignments that are equivalent to x lie in the orbit of x under G. Each orbit represents a color pattern. Thus the number of distinct patterns is equal to the number of orbits in the set X under the action of the group G. This number is given by Burnside theorem.

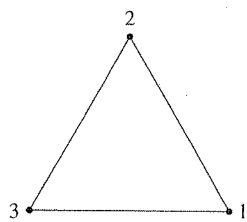
Suppose the points in the set S are coplanar. Then the group of symmetries of S is either some dihedral group

$$D_q = \{e, \alpha, \ldots, \alpha^{q-1}, \beta, \alpha\beta, \ldots, \alpha^{q-1}\beta\}$$

(where α represents a rotation through angle $2\pi/q$, and β is a reflection) or its cyclic subgroup $C_q = \{e, \alpha, \ldots, \alpha^{q-1}\}$, consisting of all rotations in D_q. If $G = D_q$, then the number of orbits in X under G gives the number of color patterns under rotations and reflections. If we wish to find the number of patterns under the stricter criterion of allowing rotations only, then we apply Burnside theorem to find the number of orbits under the group $H = C_q$ instead of under G.

In the application of Burnside theorem, we have to compute the numbers $F(g)$. For each $g \in G$, $F(g)$ is the number of color assignments that remain invariant under the action of the symmetry operation (rotation or reflection) represented by g. The identity element e in G fixes every $x \in X$, and hence $F(e) = m^n$. The following examples illustrate how we find these numbers $F(g)$ for other elements in G. The first example is the one with which we started our discussion in this chapter—namely, coloring the vertices of an equilateral triangle—but now we consider the general case of m colors.

Example 5.5.1 Each vertex of an equilateral triangle is colored by one of m given colors. Find the number of distinct patterns among all possible colorings.



Solution. Since each vertex can be colored in m ways, the total number of color assignments is m^3. The group G of symmetries of an equilateral triangle is the dihedral group of degree 3; that is,

$$G = D_3 = \{e, \alpha, \alpha^2, \beta, \alpha\beta, \alpha^2\beta\}$$

where α represents a rotation through angle $2\pi/3$, and β is a reflection in a diameter.

As mentioned above, every color assignment is invariant under the identity e; hence $F(e) = m^3$. To find the number of color assignments invariant under the other elements of G, let us number the vertices 1, 2, and 3. Then α takes vertex 1 to 2, 2 to 3, and 3 to 1. If a color assignment is invariant under α, then all three vertices must have the same color. This common color can be any one of the m given colors. Hence there are m color assignments that are invariant under α, so $F(\alpha) = m$. The same reasoning applies to α^2, so $F(\alpha^2) = m$.

Now suppose β is the reflection in the diameter passing through vertex 1. Then β takes vertex 2 to 3 and 3 to 2. If a color assignment is invariant under β, then vertices 2 and 3 must have the same color, so vertices 1 and 2 can have arbitrary colors. Hence the number of color assignments invariant under β is m^2. The same argument holds for the other two reflections $\alpha\beta$ and $\alpha^2\beta$. Hence $F(\beta) = F(\alpha\beta) = F(\alpha^2\beta) = m^2$.

By Burnside theorem, the number of patterns (that is, the number of orbits under G) is

$$k = \frac{1}{|G|}\{F(e) + F(\alpha) + F(\alpha^2) + F(\beta) + F(\alpha\beta) + F(\alpha^2\beta)\}$$

$$= \frac{1}{6}\{m^3 + 2m + 3m^2\}$$

To find the number of patterns under the stricter criterion of rotations only, we take the group of rotations $H = \{e, \alpha, \alpha^2\}$. By Burnside theorem, the number of orbits under the group H is

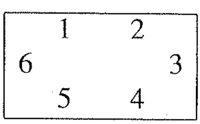
$$k' = \frac{1}{|H|}\{F(e) + F(\alpha) + F(\alpha^2)\} = \frac{1}{3}(m^3 + 2m)$$

In the particular case of only two colors, on putting $m = 2$ in the above results, we obtain

$$k = k' = 4$$

In the case $m = 3$, we have $k = 10$, $k' = 11$.

Example 5.5.2 A rectangular dining table seats six persons, two along each longer side and one on each shorter side. A colored napkin, having one of m given colors, is placed for each person. Find the number of distinct patterns among all possible color assignments.



Solution. The group of symmetries of the rectangle is

$$G = D_2 = \{e, \alpha, \beta, \alpha\beta\}$$

where α is a rotation through angle π, and β is a reflection. Let us take β to be the reflection in the line through the center parallel to the longer side of the rectangle. Then $\alpha\beta$ represents the reflection in the line parallel to the shorter side.

Let us number the six napkins as shown in the diagram above. Then α takes napkin 1 to 4, 2 to 5, 3 to 6, and vice versa. If a color assignment is invariant under the rotation α, then the napkins 1 and 4 must have the same color, 2 and 5 must have the same color, and 3 and 6 must have the same color. So we can assign arbitrary colors to napkins 1, 2, and 3. Hence the number of color assignments invariant under α is m^3.

Now β keeps napkins 3 and 6 fixed, takes 1 to 5, 2 to 4, and vice versa. If a color assignment is invariant under β, then napkins 1 and 5 must have the same color, and 2 and 4 must have the same color. So we can assign arbitrary colors to napkins 1, 2, 3, and 6. Hence the number of color assignments invariant under β is m^4. By a similar reasoning, we find that the number of color assignments invariant under $\alpha\beta$ is m^3. Therefore, by Burnside theorem, the number of patterns is

$$k = \frac{1}{|G|}\{F(e) + F(\alpha) + F(\beta) + F(\alpha\beta)\}$$
$$= \frac{1}{4}(m^6 + m^4 + 2m^3)$$

The number of patterns under the stricter criterion of rotations only is

$$k' = \frac{1}{2}\{F(e) + F(\alpha)\} = \frac{1}{2}(m^6 + m^4)$$

In the particular case of two colors, we have $k = 24,\ k' = 36$.

Example 5.5.3 (Polya's neckties) A straight necktie in the form of a long rectangular strip is divided into n bands of equal width parallel to the shorter side. Each band is colored by one of m given colors. Find the number of ties with distinct patterns.

1	2							n

Solution. The group of symmetries of a rectangle is the dihedral group D_2. But in the present case the reflection in the line parallel to the longer side doesn't play any role. The relevant group here is

$$G = D_1 = \{e, \alpha\}$$

where α may represent a rotation through angle π, or a reflection in the line through the center parallel to the shorter side of the rectangle. (The two operations are equivalent in this case.)

If a color assignment is invariant under α, then the bands 1 and n must have the same color, the bands 2 and $n-1$ must have the same color, and so on. In general, the bands i and $n+1-i$ must have the same color. If n is even, we can assign arbitrary colors to bands $1, \ldots, \frac{n}{2}$; hence $F(\alpha) = m^{n/2}$. But if n is odd, the bands $1, \ldots, \frac{n+1}{2}$ can be assigned arbitrary colors. (The $\frac{n+1}{2}$th band is the band in the middle.) Hence $F(\alpha) = m^{(n+1)/2}$.

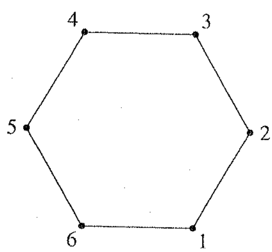
Therefore, by Burnside theorem, the number of patterns is

$$k = \frac{1}{|G|}\{F(e) + F(\alpha)\}$$

$$= \begin{cases} \frac{1}{2}(m^n + m^{n/2}) & \text{if } n \text{ is even} \\ \frac{1}{2}(m^n + m^{(n+1)/2}) & \text{if } n \text{ is odd} \end{cases}$$

For example, if $m=2$ and $n=8$, then $k=\frac{1}{2}(2^8+2^4)=136$. If $m=2$ and $n=9$, then $k=\frac{1}{2}(2^9+2^5)=272$.

Example 5.5.4 Suppose each vertex of a regular hexagon is colored by one of m given colors. Find the number of distinct patterns among all colorings.



Solution. The group of symmetries of a regular hexagon is

$$G = D_6 = \{e, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \beta, \alpha\beta, \alpha^2\beta, \alpha^3\beta, \alpha^4\beta, \alpha^5\beta\}$$

where α represents a rotation through angle $\pi/3$, and β is a reflection in a diameter. Let us number the vertices, taken in order, as 1, 2, 3, 4, 5, and 6. If a color assignment is invariant under the rotation α, then each vertex must have the same color; hence $F(\alpha)=m$. If a color assignment is invariant under α^2, then vertices 1, 3, and 5 have the same color, and vertices 2, 4, and 6 have the same color. Hence $F(\alpha^2)=m^2$. If a color assignment is invariant under α^3, then vertices 1 and 4 have the same color, 2 and 5 have the same color, and 3 and 6 have the same color. Hence $F(\alpha^3)=m^3$. Similarly, we find $F(\alpha^4)=m^2$ and $F(\alpha^5)=m$.

Suppose β represents reflection in the diameter through vertex 1. If a color assignment is invariant under the reflection β, then the vertices 2 and 6 have the same color and 3 and 5 have the same color. So we can assign arbitrary colors to vertices 1, 2, 3 and 4. Hence $F(\beta)=m^4$.

Now $\alpha\beta$ is a reflection in the diameter passing through the middle point between vertices 1 and 2. If a color assignment is invariant under $\alpha\beta$, then vertices 1 and 2 must have the same color, 3 and 6 have the same color, and 4 and 5 have the same color. Hence $F(\alpha\beta) = m^3$.

By similar arguments, we obtain $F(\alpha^2\beta) = F(\alpha^4\beta) = m^4$ and $F(\alpha^3\beta) = F(\alpha^5\beta) = m^3$. Hence, by Burnside theorem, the number of patterns is

$$k = \frac{1}{|G|}\left\{\begin{array}{l} F(e) + F(\alpha) + F(\alpha^2) + F(\alpha^3) + F(\alpha^4) + F(\alpha^5) \\ + F(\beta) + F(\alpha\beta) + F(\alpha^2\beta) + F(\alpha^3\beta) + F(\alpha^4\beta) + F(\alpha^5\beta) \end{array}\right\}$$

$$= \frac{1}{12}(m^6 + 3m^4 + 4m^3 + 2m^2 + 2m)$$

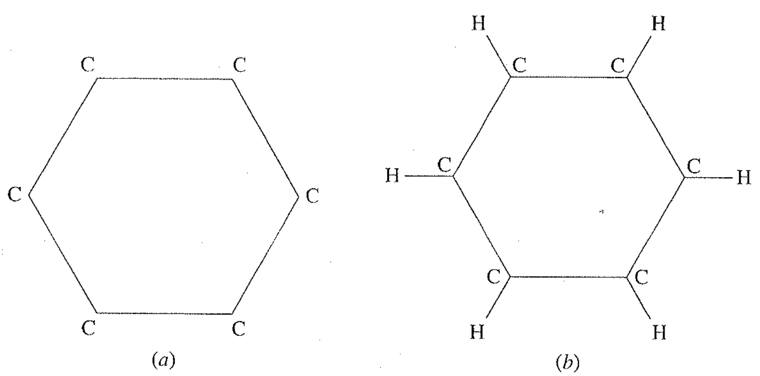
Under the stricter criterion of rotations only, the number of patterns is

$$k' = \frac{1}{6}\{F(e) + F(\alpha) + F(\alpha^2) + F(\alpha^3) + F(\alpha^4) + F(\alpha^5)\}$$

$$= \frac{1}{6}(m^6 + m^3 + 2m^2 + 2m)$$

In the particular case $m = 2$, we have $k = 13$, $k' = 14$.

Example 5.5.4 has an important application in chemistry. From the carbon ring consisting of six carbon atoms (figure a) one can obtain several chemically different molecules by attaching to each carbon atom either a hydrogen atom H or the group CH_3. For example, if a hydrogen atom is attached to each carbon atom, the result is a molecule of benzene (figure b). The question is: How many chemically different molecules can be obtained in this manner?



It is obvious that the problem is mathematically the same as finding the number of patterns in coloring the vertices of a regular hexagon with two colors. This problem was solved in Example 5.5.4. There are 13 chemically different molecules that can be obtained from the carbon ring.

From the examples above, we see that the main work involved in the application of Burnside theorem is the computation of the numbers $F(g)$. Now we obtain a modified version of Burnside theorem that makes this task more systematic and somewhat easier.

As before, let S denote the set of points to be colored, and let C denote the set of colors. Then any coloring of the points in S with colors from the set C is in fact a mapping from S to C, so the set X of all color assignments is the set of all mappings from S to C, which we write as $X = C^S$. Let G be a group of symmetries of the set S. Then G acts naturally on the set S. Geometrically, it is evident that G also acts on the set X, a fact we have used in the examples above. But let us now give a general algebraic proof of this property.

Let G be a group acting on a set S, let C be an arbitrary nonempty set, and let $X = C^S$ be the set of all mappings from S to C. The action of G on S induces an action of G on X as follows: For any $g \in G$, $f \in X$, we define $g * f \in X$ as the mapping $g * f : S \to C$ with

$$(g * f)(s) = f(g^{-1}s)$$

for all $s \in S$. Then $e * f(s) = f(e^{-1}s) = f(s)$ for all $s \in S$; hence $e * f = f$. Further, for all $g, h \in G$ and $f \in X$,

$$((gh) * f)(s) = f((gh)^{-1}s) = f(h^{-1}g^{-1}s) = (h * f)(g^{-1}s) = (g * (h * f))(s)$$

holds for all $s \in S$. Hence $(gh) * f = (g * (h * f))$. This proves that $*$ is an action of G on X.

Recall that each element $g \in G$ induces a permutation σ_g of the set S given by the rule $\sigma_g(s) = gs$ for all $s \in S$.

THEOREM 5.5.5 Let G be a finite group acting on a finite set S, let C be a finite set of m elements, and let $X = C^S$ be the set of all mappings from S to C. Then the number of elements in X fixed by $g \in G$ is

$$F(g) = m^{\lambda(g)}$$

where $\lambda(g)$ is the number of disjoint cycles (including cycles of length 1) in the cycle decomposition of the permutation σ_g of S induced by g.

Consequently, the number k of orbits in X under the action of G is given by

$$k = \frac{1}{|G|} \sum_{g \in G} m^{\lambda(g)}$$

Proof: Let $g \in G$ and $f \in X$. If $g * f = f$, then $f(s) = (g * f)(s) = f(g^{-1}s)$ for all $s \in S$. Hence $f(gs) = f(g^{-1}gs) = f(s)$ for all $s \in S$. Conversely, if $f(gs) = f(s)$ for all $s \in S$, then $(g * f)(s) = f(g^{-1}s) = f(gg^{-1}s) = f(s)$ for all $s \in S$; hence $g * f = f$. Thus $f \in Fix(g)$ if and only if $f(gs) = f(s)$ for all $s \in S$.

Let σ_g be the permutation of S determined by g; that is, $\sigma_g(s) = gs$ for all $s \in S$. Let $\sigma_g = \gamma_1 \cdots \gamma_\lambda$ be the decomposition of σ_g into disjoint cycles (including cycles of length 1). Any cycle in this decomposition is of the form

$$\gamma = (a \; ga \; g^2a \ldots g^{r-1}a)$$

If $f \in Fix(g)$, then $f(a) = f(ga) = \cdots = f(g^{r-1}a)$; hence f is constant on the elements in the cycle γ. This holds for every cycle γ_i in the decomposition of σ_g.

Conversely, if f is constant on every cycle γ_i, then $f(gs) = f(s)$ for all $s \in S$. Hence $f \in Fix(g)$ if and only if f is constant on each cycle in the decomposition of σ_g.

Let $f \in Fix(g)$, and let $f_1, \ldots, f_\lambda \in C$ be the values of f on the cycles $\gamma_1, \ldots, \gamma_\lambda$, respectively. Then $f_1, \ldots, f_\lambda$ can be each chosen in m ways. Hence there are exactly m^λ elements in $Fix(g)$. This proves the first part of the theorem.

Hence, by Burnside theorem,

$$k = \frac{1}{|G|} \sum_{g \in G} F(g) = \frac{1}{|G|} \sum_{g \in G} m^{\lambda(g)}$$ ∎

To illustrate the use of Theorem 5.5.5, we work out again the problem of Example 5.5.2 by the new method.

Example 5.5.6 Do the problem of Example 5.5.2 by using Theorem 5.5.5.

Solution. With the notation of Example 5.5.2, we identify each element in the group G with the permutation induced by it on the set $\{1, 2, 3, 4, 5, 6\}$ and find its cycle decomposition:

$$e = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} = (1)(2)(3)(4)(5)(6)$$

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{pmatrix} = (1\,4)(2\,5)(3\,6)$$

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 3 & 2 & 1 & 6 \end{pmatrix} = (1\,5)(2\,4)(3)(6)$$

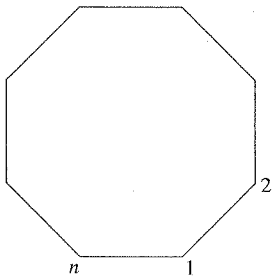
$$\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 6 & 5 & 4 & 3 \end{pmatrix} = (1\,2)(3\,6)(4\,5)$$

Hence $\lambda(e) = 6$, $\lambda(\alpha) = 3$, $\lambda(\beta) = 4$, $\lambda(\alpha\beta) = 3$. Therefore, by Theorem 5.5.5,

$$k = \frac{1}{4}(m^6 + m^4 + 2m^3)$$

In the following example we solve the general problem of which Examples 5.5.1 and 5.5.4 are special cases.

Example 5.5.7 Suppose each vertex of a regular polygon of n sides is colored by one of m given colors. Find the number of distinct patterns among all colorings.



Solution. The group G of symmetries of a regular polygon of n sides is the dihedral group of degree n,

$$G = D_n = \{e, \alpha, \dots, \alpha^{n-1}, \beta, \alpha\beta, \dots, \alpha^{n-1}\beta\}$$

where α is a rotation through angle $2\pi/n$ and β is reflection in a diameter, say through the vertex 1. Interpreting the elements of G as permutations of the set $S = \{1, \dots, n\}$ of the vertices, we have

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 2 & 3 & 4 & \dots & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 1 & n & n-1 & \dots & 2 \end{pmatrix}$$

Consider first the set $H = \{e, \alpha, \dots, \alpha^{n-1}\}$ of rotations in G. H is a cyclic subgroup of G of order n. Let g be an element of order r in the group H. We show that the number of cycles in the decomposition of the permutation g is $\lambda(g) = n/r$. Since $g^r = e$, we have $g^r(i) = i$ for each $i = 1, \dots, n$. We claim that given any i, r is the least positive integer such that $g^r(i) = i$. Suppose there exists a positive integer $s < r$ such that $g^s(j) = j$ for some j. Now g^s is a rotation of the polygon. If it takes j to j, it must take each i to i, so $g^s(i) = i$ for all $i = 1, \dots, n$. This means $g^s = e$, which contradicts the fact that g is of order r. Therefore given any $i \in \{1, \dots, n\}$, i generates the cycle $(i\ g(i) \dots g^{r-1}(i))$ of length r. Hence each cycle in the cyclic decomposition of the permutation g is of length r. It follows that the number of cycles is $\lambda(g) = n/r$.

Since H is a cyclic group of order n, the order of every element in H is a divisor of n. Moreover, given any divisor r of n, the number of elements in H of order r is $\varphi(r)$, where $\varphi(r)$ denotes the number of positive integers less than r and relatively prime to r. Hence the contribution to the summation $\sum m^{\lambda(g)}$ (in Theorem 5.5.5) from the rotations in the group G is

$$\sum_{g \in H} m^{\lambda(g)} = \sum_{r|n} \varphi(r)\, m^{n/r}$$

Now consider the set K of elements other than rotations in G; that is, $K = G - H = \{\beta, \alpha\beta, \dots, \alpha^{n-1}\beta\}$. Here two cases arise.

1. If n is odd, then each element g in K represents a reflection in a diameter through some vertex j. So g fixes the vertex j and interchanges vertices $j+i$ and $j-i$ for $i=1,\ldots,\frac{n-1}{2}$. Thus the cyclic decomposition of g consists of one cycle (j) of length 1 and $\frac{n-1}{2}$ cycles $(j-1\ j+1)$ of length 2. Hence $\lambda(g)=\frac{n+1}{2}$.
2. Suppose n is even. Then exactly half of the elements in K represent reflections in a diameter passing through two opposite vertices, and the remaining $\frac{n}{2}$ elements represent reflections in a diameter passing through the middle points of two opposite sides. It is easily seen that if g is a reflection of the first type, then $\lambda(g)=\frac{n+2}{2}$. If g is of the second type, then $\lambda(g)=\frac{n}{2}$.

Hence the contribution to the summation $\sum m^{\lambda(g)}$ from the reflections in the group G is

$$\sum_{g\in K} m^{\lambda(g)} = \begin{cases} nm^{(n+1)/2} & \text{if } n \text{ is odd} \\ \frac{n}{2}m^{(n+2)/2}+\frac{n}{2}m^{n/2} & \text{if } n \text{ is even} \end{cases}$$

If we combine the contributions from H and K, the number k of patterns is given by

$$k=\frac{1}{|G|}\sum_{g\in G} m^{\lambda(g)}$$

$$= \begin{cases} \frac{1}{2n}\left(\sum_{r|n}\varphi(r)\,m^{n/r}+nm^{(n+1)/2}\right) & \text{if } n \text{ is odd} \\ \frac{1}{2n}\left(\sum_{r|n}\varphi(r)\,m^{n/r}+\frac{n}{2}(m^{(n+2)/2}+m^{n/2})\right) & \text{if } n \text{ is even} \end{cases}$$

Under the stricter criterion of rotations only, the number of patterns is

$$k'=\frac{1}{|H|}\sum_{g\in H} m^{\lambda(g)}=\frac{1}{n}\sum_{r|n}\varphi(r)\,m^{n/r}$$

EXERCISES 5.5

1. Find (without using the result of Example 5.5.7) the number of patterns obtained on coloring the vertices of a square with m colors.
2. Repeat Exercise 1 for a regular pentagon.
3. Repeat Exercise 1 for a rectangle.
4. Find the number of distinct necklaces with p beads (p prime), where each bead can have any one of n colors.

5. Find the number of distinct bracelets of six beads, where each bead is red, blue, or white.
6. A rectangular design consists of 11 parallel stripes of equal width. If each stripe can be painted red, blue, or green, find the number of possible patterns.
7. Each side of an equilateral triangle is divided into two equal parts, and each part is colored red or green. Find the number of patterns.
8. Each side of a square is divided into three equal parts, and each part is colored red, yellow, or green. Find the number of patterns.
9. Each side of a regular polygon of n sides is divided into q equal parts, and each part is painted with one of m colors. Find the number of patterns.
10. Each vertex of an equilateral triangle is colored with one of four colors such that at least two vertices have different colors. Find the number of patterns.
11. Repeat Exercise 10 for a square.
12. The interior of an equilateral triangle is divided into six parts by the medians. Each part is painted with one of m colors. Find the number of patterns.
13. The sides of a rectangle are 3 feet and 4 feet long. The rectangle is divided into 12 equal squares, and each square is painted with one of m colors. Find the number of patterns.
14. Repeat Exercise 13 for a rectangle with sides of lengths 4 feet and 6 feet, divided into 24 squares.
15. Repeat Exercise 13 for a rectangle with sides of lengths 5 feet and 7 feet, divided into 35 squares.
16. Repeat Exercise 13 for a rectangle with sides of lengths p feet and q feet, divided into pq squares.
17. Find the number of ways in which the faces of a regular tetrahedron can be painted with m colors.
18. Find the number of ways in which the faces of a cube can be painted with m colors.

5.6 POLYA'S THEOREM AND PATTERN INVENTORY

We now consider the problem of finding the number of color patterns in which the colors occur with preassigned frequencies. For instance, we found in Example 5.5.2 the number of distinct patterns in m-colorings of six napkins arranged on a rectangular table. We may now ask the question: What is the number of patterns in which there are exactly one yellow, two red, and three green napkins? It is questions of this type that can be answered by using Polya's theorem, which we are about to prove now.